Navier–Stokes in the HRT–SIGAP Spectral Framework: A Program and Conjectures

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This paper proposes a structured research program connecting the three-dimensional incompressible Navier–Stokes equations to the spectral geometry of the Holographic Ring Theory (HRT) and the Spectral Information Geometric Action Principle (SIGAP).

Instead of claiming a proof of global regularity, this work reframes previous HRT-based arguments as a sequence of conjectures and analytic tasks: (i) embedding fluid fields into a SIGAP-compatible Hilbert space; (ii) expressing classical energy and enstrophy inequalities in terms of toroidal spectral coefficients; (iii) exploring whether polynomial constraints defining the ideal $I_{\mathrm{RING}} \subset \mathbb{Z}[\pi]$ can impose effective bounds on nonlinear mode interactions; and (iv) identifying whether SIGAP geometric structure can constrain high-frequency energy cascade in turbulence.

Two conjectures (Spectral Growth Bound and Toroidal Enstrophy Control) are formulated from the heuristic estimates originally motivated by HRT. These conjectures, if validated, could yield new pathways toward regularity criteria. The result is a mathematically grounded roadmap linking modern spectral geometry with classical fluid analysis.

I. INTRODUCTION

The three-dimensional incompressible Navier–Stokes equations,

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

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govern the motion of viscous, incompressible fluids. The Clay Millennium Problem asks whether smooth initial data $u_0 \in C^\infty(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ leads to global smooth solutions, or whether finite-time singularities can form.

Holographic Ring Theory (HRT) proposes that a wide range of physical and number-theoretic structures are encoded in the polynomial ideal

$$I_{\mathrm{RING}} = \left\langle 4\pi^3 + \pi^2 + \pi - 137,\ \pi^4 - 97,\ m^2 + n^2 - \pi^2,\ mn - \pi \right\rangle \subset \mathbb{Z}[\pi]. \tag{3}$$

The Spectral Information Geometric Action Principle (SIGAP) provides a geometric operator whose eigenvalue spectrum reproduces the generators of this ideal.

This paper does *not* claim that the HRT structure resolves the Navier–Stokes regularity problem. Instead, it reinterprets the algebraic-spectral arguments as a rigorous program composed of analytic conjectures and operator-theoretic questions. The resulting framework may open new directions by combining toroidal spectral geometry with classical PDE methods.

II. HRT–SIGAP SPECTRAL FOUNDATIONS

SIGAP introduces a unified geometric operator acting on a multiplet $\Psi = (\Phi, I)^T$, schematically of the form

$$O_{\mathrm{geom}}\Psi = \partial_\Psi U_{\mathrm{unif}} + J, \tag{4}$$

whose spectrum encodes dispersion, RG flow, zeta structure, and the HRT ideal. In this setting, I_{RING} is realized as the spectral determinant of the operator.

Although Navier–Stokes does not arise directly from SIGAP, one may attempt to embed fluid variables into a SIGAP-compatible Hilbert space and analyze the effect of the toroidal spectral structure on nonlinear mode interactions.

III. TOROIDAL SPECTRAL REPRESENTATION OF FLUID FIELDS

Let $u(x,t)$ be a divergence-free velocity field. We introduce a toroidal spectral expansion,

$$u(x,t) = \sum_{(m,n)\in S} a_{m,n}(t)\, u_{m,n}(x), \tag{5}$$

where S is a subset of integer pairs indexing toroidal winding modes suggested by the SIGAP/HRT geometry.

Likewise, the vorticity field is expanded as

$$\omega(x,t) = \sum_{(m,n)\in S} a_{m,n}(t)\,\omega_{m,n}(x). \tag{6}$$

This representation is not claimed to coincide with classical Laplacian eigenmodes on $\mathbb{T}^3$. Instead, it functions as a surrogate basis for expressing nonlinear interactions in a framework compatible with the I_{RING} spectral structure.

IV. CLASSICAL NAVIER–STOKES ESTIMATES IN SPECTRAL FORM

The standard energy identity for Navier–Stokes on $\mathbb{T}^3$ or $\mathbb{R}^3$ is

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \nu\|\nabla u(t)\|_{L^2}^2 = \langle f, u(t)\rangle. \tag{7}$$

In spectral variables,

$$\|u(t)\|_{H^s}^2 = \sum_{m,n}(1+m^2+n^2)^s |a_{m,n}(t)|^2, \tag{8}$$

and the nonlinear term becomes

$$\langle u_{m,n}, (u\cdot\nabla)u\rangle\,. \tag{9}$$

The analytic difficulty is to bound these coefficients. The HRT structure suggests relations among (m,n) via I_{RING}, leading to heuristic bounds in earlier work. Here, these ideas are promoted to formal conjectures.

V. CONJECTURE 1: SPECTRAL GROWTH BOUND

Conjecture 1. *(HRT Spectral Growth Bound)* There exists a function $K(C_{\mathrm{spec}})$, depending on spectral properties associated with I_{RING}, such that

$$|\langle u_{m,n}, (u\cdot\nabla)u\rangle| \le K(C_{\mathrm{spec}})\,\Phi(\|u\|_{H^1}), \tag{10}$$

for some growth function Φ.

If such a bound holds uniformly over all (m,n), then an energy inequality of the form

$$\|u(t)\|_{H^s}^2 \le C(s)\|u_0\|_{H^s}^2 \exp\!\left(\frac{C_{\mathrm{spec}}}{\nu}t\right) \tag{11}$$

would follow.

VI. CONJECTURE 2: TOROIDAL ENSTROPHY CONTROL

The classical enstrophy satisfies

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + 2\nu\|\nabla\omega\|_{L^2}^2 \leq C\|u\|_{L^4}^4. \tag{12}$$

Conjecture 2. *(Toroidal Enstrophy Control)* If the SIGAP/HRT spectral structure constrains high-frequency interactions, then one may obtain a bound of the form

$$\|\omega(t)\|_{L^2}^2 \leq \|\omega_0\|_{L^2}^2 + \frac{C_{\text{spec}}}{\nu^2}\int_0^t \|u(s)\|_{L^2}^4\, ds, \tag{13}$$

where C_{spec} reflects spectral data associated with I_{RING}.

VII. PROGRAM ROADMAP

To evaluate these conjectures, the following analytic tasks are identified:

1. **Construct a SIGAP-compatible basis** for divergence-free fields.

2. **Establish product estimates** for nonlinear terms in the toroidal basis.

3. **Characterize the SIGAP spectral gap** and its relation to integer-mode structure.

4. **Compare HRT constraints with classical regularity criteria**, such as Prodi–Serrin or Beale–Kato–Majda.

5. **Perform computational experiments** to test approximate bounds for small sets of modes.

VIII. LOW-DIMENSIONAL CONSISTENCY CHECKS

For small winding pairs (m, n), the toroidal eigenvalue structure implied by I_{RING} may be compared with classical Fourier-mode behavior. These calculations, while not proving any regularity results, help calibrate the spectral dictionary between HRT geometry and fluid modes.

IX. CONDITIONAL GLOBAL REGULARITY THEOREM

We now state a precise conditional result showing that the spectral bounds proposed in Conjectures 1 and 2 are sufficient to guarantee global regularity of the three-dimensional incompressible Navier–Stokes equations. This theorem is expressed entirely within standard PDE function spaces and does not assume any particular structure of the I_{RING} ideal, except insofar as it provides the constants appearing in the conjectures.

A. Hypotheses

Let u be a Leray–Hopf weak solution of the Navier–Stokes equations on $\mathbb{T}^3$ or $\mathbb{R}^3$ with smooth initial data. We assume the following three hypotheses:

(H1) SIGAP–HRT Spectral Representation. There exists a complete, divergence-free orthonormal basis $\{u_{m,n}\}_{(m,n)\in S}$ of L^2_σ such that

$$u(x,t) = \sum_{(m,n)\in S} a_{m,n}(t)\, u_{m,n}(x), \tag{14}$$

with convergence in H^s for every $s \geq 0$. No assumption is made that $u_{m,n}$ are Laplacian eigenfunctions.

(H2) Spectral Growth Bound (Conjecture 1). There exists a constant $C_{\mathrm{spec}} > 0$ such that for all $(m,n) \in S$,

$$|\langle u_{m,n},\, (u\cdot\nabla)u\rangle| \leq C_{\mathrm{spec}}\, \Phi(\|u\|_{H^1}), \tag{15}$$

where Φ is a continuous, nondecreasing function.

(H3) Toroidal Enstrophy Control (Conjecture 2). For vorticity $\omega = \nabla \times u$,

$$\|\omega(t)\|^2_{L^2} \leq \|\omega_0\|^2_{L^2} + \frac{C_{\mathrm{spec}}}{\nu^2} \int_0^t \|u(s)\|^4_{L^2}\, ds. \tag{16}$$

B. Main Conditional Theorem

Theorem (Conditional Global Regularity). Assume hypotheses (H1)–(H3) hold. Then every Leray–Hopf weak solution u with smooth, divergence-free initial data is smooth for all $t > 0$. In particular, $u \in C^\infty([0,\infty)\times\mathbb{R}^3)$.

Sketch of Proof. The proof follows classical ideas, modified to incorporate the spectral bounds. The energy inequality provides global control of $\|u(t)\|_{L^2}$. Hypothesis (H3) then implies $\|\omega(t)\|_{L^2}$ is bounded on every finite interval. By the standard identity $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$ for divergence-free fields, we obtain uniform control of the H^1 norm.

Next, hypothesis (H2) bounds the nonlinear term in the H^s energy inequality:

$$\frac{d}{dt}\|u(t)\|_{H^s}^2 = -2\nu\|\nabla^{s+1}u(t)\|_{L^2}^2 + 2\sum_{m,n}(1+m^2+n^2)^s\, a_{m,n}\,\langle u_{m,n},\,(u\cdot\nabla)u\rangle\,. \tag{17}$$

Using (H2) and boundedness of H^1, we obtain

$$\frac{d}{dt}\|u(t)\|_{H^s}^2 \le C_{\text{spec}}\,\Psi(\|u(t)\|_{H^s}), \tag{18}$$

for a continuous, nondecreasing Ψ. Gronwall's inequality then yields a finite bound on $\|u(t)\|_{H^s}$ for all finite times and for all $s \ge 1$. Bootstrapping gives smoothness.

□

C. Interpretation

This theorem shows that the HRT–SIGAP spectral framework offers a valid *pathway* to resolving the Navier–Stokes problem: if the nonlinear spectral interactions obey the bounds predicted by I_{RING}, then global regularity follows by standard PDE methods.

Thus, the essential question is not whether Navier–Stokes can be made smooth, but whether the SIGAP/HRT spectral geometry can be shown to imply the key bounds in (H2) and (H3).

X. CONCLUSION

This paper reformulates previous HRT-based arguments as a coherent program for investigating Navier–Stokes dynamics through SIGAP spectral geometry. The conjectures presented here represent the mathematically meaningful content of earlier heuristic estimates. Whether the I_{RING} structure can truly constrain turbulent energy cascade remains an open question, but the HRT–SIGAP framework offers a novel pathway for exploration.

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[1] B. Jarvis, *Holographic Ring Theory*, Vor-Techs Research and Development (2024).

[2] B. Jarvis, *Spectral Information Geometric Action Principle*, Vor-Techs Research and Development (2025).

[3] Clay Mathematics Institute, *Navier–Stokes Equations, Official Problem Statement* (2000).

Navier–Stokes Regularity via HRT–SIGAP Spectral Geometry: A Program and Mechanism

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This work proposes a spectral–geometric mechanism for taming the nonlinear term in the three-dimensional incompressible Navier–Stokes equations. Inspired by the Holographic Ring Theory (HRT) and the Spectral Information Geometric Action Principle (SIGAP), we introduce an algebraically thin spectral subset of $\mathbb{Z}^3$ and a SIGAP-induced dispersive frequency assignment. A detailed infinite-dimensional toy model demonstrates how these structures generate triad sparsity and non-resonant suppression, suggesting a path toward a uniform spectral bound on the nonlinear term. A set of conjectures outlines how these mechanisms may extend to the full Navier–Stokes system.

I. INTRODUCTION

1. The Millennium Problem. The three–dimensional incompressible Navier–Stokes equations describe the dynamics of viscous fluid motion and sit at the foundation of modern mathematical physics. For smooth, divergence–free initial data $u_0 \in H^1(\mathbb{T}^3)$, the Cauchy problem

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u &= -\nabla p + \nu \Delta u, \\ \nabla \cdot u &= 0, \\ u(\cdot, 0) &= u_0, \end{aligned}$$

is known to possess local solutions. The Clay Millennium Problem asks whether such solutions can *blow up* in finite time, or whether they remain *globally smooth* for all $t > 0$. Despite remarkable progress on related models, the general question of global regularity for Navier–Stokes in three dimensions remains unresolved.

2. The spectral challenge. A central difficulty lies in the structure of the *quadratic nonlinearity* and the associated *triad interactions* in Fourier space. Writing the velocity field in Fourier modes,

$$u(x,t) = \sum_{k \in \mathbb{Z}^3} a_k(t)\, e^{ik\cdot x},$$

the nonlinear term couples modes through triads $k = p + q$, yielding an infinite network of interactions. This triad graph is *dense*, and although dissipation controls high frequencies, the nonlinearity can in principle transfer energy toward small scales rapidly enough to generate singularities. Understanding the structure of these interactions—and in particular, the role of *resonant* and *near–resonant* triads—has become a major theme in modern analysis.

3. Spectral thinning as a mechanism. One promising direction is to identify structural conditions under which the nonlinear term is tamed by geometry rather than by brute–force estimates. If the triad graph were sufficiently *sparse*, or if a mechanism acted to suppress non–resonant interactions so that only a limited family of couplings survived, then the nonlinear term might admit a *uniform spectral bound*

$$\big|\langle u_k, (u \cdot \nabla)u \rangle\big| \;\le\; C_{\mathrm{spec}}\, \Phi(\|u\|_{H^1}),$$

for a mode–independent constant C_{spec}. Such a bound would fundamentally change the analytic landscape of Navier–Stokes by altering possible cascade scenarios and providing improved control on high–frequency modes. However, no natural mechanism producing such spectral sparsity is known in the classical formulation.

4. The HRT–SIGAP framework. This paper explores a novel route toward spectral thinning inspired by the *Holographic Ring Theory* (HRT) and the *Spectral Information Geometric Action Principle* (SIGAP), structures

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originally developed in a different physical context. In these models, spectral data obey algebraic relations encoded in an ideal

$$I_{\mathrm{RING}} = \langle \text{polynomial relations among } (\pi, m, n), \dots \rangle,$$

which, when projected to the integer lattice, produces an *infinite but extremely sparse* set of lattice points organized along algebraic curves. This algebraic sparsity motivates the hypothesis that certain classes of Fourier modes in fluid flow might effectively lie on—or be attracted toward—similarly thin spectral subsets.

The SIGAP operator further induces a nontrivial *dispersive frequency* ω_k on each mode. Such dispersion is well known to suppress non–resonant interactions by introducing rapid oscillations in the interaction picture. Combining this effect with algebraic sparsity yields a powerful *two–tier mechanism*:

(i) HRT spectral geometry thins the triad graph by restricting viable horizontal modes;

(ii) SIGAP dispersion suppresses most remaining couplings through non–resonant oscillatory cancellation.

Together, these structures suggest a path toward the type of uniform spectral bound central to controlling the nonlinear term.

5. Purpose of this work. The aim of this paper is not to claim a complete solution to the Navier–Stokes regularity problem, but to develop a *precise mathematical program* that leverages HRT–SIGAP spectral geometry to formulate a new mechanism for regularity. We introduce:

- a concrete HRT–algebraic spectral subset of $\mathbb{Z}^3$,
- a SIGAP–inspired dispersive frequency assignment,
- a fully analyzed infinite–dimensional toy model demonstrating global regularity under these constraints, and
- a set of conjectures describing how these mechanisms may extend to the full Navier–Stokes system.

The results suggest that spectral thinning combined with dispersive suppression may constitute a viable analytic path toward global regularity. The remainder of the paper develops this program in detail.

II. BACKGROUND: NAVIER–STOKES IN SPECTRAL FORM

The incompressible Navier–Stokes equations on the three–dimensional torus $\mathbb{T}^3 = [0, 2\pi]^3$ may be written as

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \qquad \nabla \cdot u = 0, \tag{1}$$

where $u(x,t)$ is the velocity field, $p(x,t)$ is the pressure, and $\nu > 0$ denotes kinematic viscosity. We recall the standard Fourier–spectral formulation used throughout this paper.

A. Fourier expansion and divergence–free projection

For $k \in \mathbb{Z}^3$, let $e_k(x) = e^{ik\cdot x}$ and expand

$$u(x,t) = \sum_{k\in\mathbb{Z}^3} a_k(t)\, e_k(x), \qquad p(x,t) = \sum_{k\in\mathbb{Z}^3} p_k(t)\, e_k(x),$$

where $a_k(t) \in \mathbb{C}^3$ satisfies the reality condition $a_{-k} = \overline{a_k}$. The incompressibility constraint $\nabla \cdot u = 0$ becomes

$$k \cdot a_k(t) = 0.$$

Let P_k denote the Leray projection onto the plane orthogonal to k, i.e.

$$P_k v = v - \frac{k(k\cdot v)}{|k|^2}, \qquad v \in \mathbb{C}^3.$$

Applying P_k to the Fourier transform of (1) eliminates the pressure.

B. Spectral evolution equation

The resulting evolution equation for the Fourier coefficients is

$$\frac{d}{dt}a_k(t) = -\nu|k|^2 a_k(t) - i\sum_{p+q=k}(k\cdot a_p(t))\,P_k a_q(t). \tag{2}$$

The sum is taken over all ordered pairs $(p,q)\in\mathbb{Z}^3\times\mathbb{Z}^3$ satisfying $p+q=k$, and the term P_k ensures that $a_k(t)$ remains divergence–free.

It is often convenient to write the quadratic nonlinearity as

$$\mathcal{N}_k(a) = \sum_{p+q=k} B(k,p,q)\,a_p a_q, \tag{3}$$

where $B(k,p,q)$ is a trilinear tensor determined by the Leray projection and the bilinear form $(u\cdot\nabla)u$. The precise form of B is well understood but not essential here; what matters is the *triad structure* enforced by $p+q=k$.

C. Triad interactions and resonance

Each mode a_k interacts only with pairs (a_p, a_q) whose wavevectors form a *triad*:

$$k = p+q.$$

This constraint generates an infinite graph on $\mathbb{Z}^3$ whose vertices are wavevectors and whose edges correspond to nonlinear couplings. The density of this triad graph underlies many difficulties in understanding the flow of energy across scales.

Given a linear dispersion relation ω_k (such as in rotating or stratified flows), one may study *resonant* and *non-resonant* interactions via the phase

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q.$$

A triad is said to be *resonant* if $\Omega_{k,p,q}=0$. Resonant interactions can accumulate energy over long times, while non–resonant terms often admit cancellations due to oscillatory phases. In the classical incompressible Navier–Stokes equations, however, no natural dispersion relation is present, and the structure of triads remains fully dense.

D. Energy balance and challenges for regularity

The kinetic energy

$$E(t) = \frac{1}{2}\sum_{k\in\mathbb{Z}^3}|a_k(t)|^2$$

satisfies

$$\frac{d}{dt}E(t) = -\nu\sum_k |k|^2|a_k(t)|^2,$$

since the nonlinearity is energy–preserving. Although dissipation controls sufficiently high frequencies, the quadratic nonlinearity can transfer energy to small scales, potentially fast enough to outpace dissipation.

The central difficulty in proving global regularity is that the nonlinear term in (2) does not admit a uniform bound of the form

$$|\mathcal{N}_k(a)| \;\le\; C_{\mathrm{spec}}\,\Phi(\|a\|_{H^1}),$$

with C_{spec} independent of k. A mechanism capable of enforcing such a bound—for example, via structural sparsity of triads or suppression of non–resonant interactions—would significantly change the analytic picture. This concept motivates the spectral–geometric program developed in the following sections.

III. THE HRT–SIGAP SPECTRAL FRAMEWORK: CONCEPTUAL OVERVIEW

The Holographic Ring Theory (HRT) and the Spectral Information Geometric Action Principle (SIGAP) originate in a geometric and algebraic program distinct from fluid dynamics, but they possess a spectral structure that suggests a novel route toward controlling nonlinear interactions in Navier–Stokes. In this section we provide a minimal, self-contained overview of the aspects relevant to the present work. The goal is not to review the full framework, but to highlight two structural features:

(i) the emergence of extremely *thin algebraic subsets* of integer lattice points, and

(ii) a natural *dispersive frequency assignment* for modes, induced by SIGAP.

These two ingredients—algebraic sparsity and dispersion—form the foundation of the spectral mechanism developed in later sections.

A. Algebraic spectral relations and the ideal I_{RING}

In the HRT framework, spectral variables such as (m, n) and geometric constants (such as π or derived constants) satisfy algebraic relations encoded in a polynomial ideal

$$I_{\mathrm{RING}} \subset \mathbb{Z}[\pi, m, n, \dots].$$

The explicit generators of this ideal are not needed here; what matters is that the relations include constraints of the schematic form

$$m^2 + n^2 = \pi^2, \qquad mn = \pi, \tag{4}$$

together with higher-degree relations coupling π to other geometric quantities.

Since π is irrational, the system (4) has no exact integer solutions. However, by replacing π with rational convergents p_k/q_k from its continued fraction expansion and multiplying through by q_k^2, one obtains *exact* Diophantine systems of the type

$$x^2 + y^2 = p_k^2, \qquad xy = p_k q_k, \tag{5}$$

with $(x, y) \in \mathbb{Z}^2$. Each integer p_k defines a "shell" consisting of the integer points on the circle $x^2 + y^2 = p_k^2$ that simultaneously satisfy the hyperbolic constraint $xy = p_k q_k$.

The crucial observation is that each such shell contains only *finitely many* integer points—often only a handful—and as k varies one obtains an *infinite but extremely sparse* subset

$$\mathcal{H}_\pi \;=\; \bigcup_{k\geq 1} \left\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = p_k^2,\; xy = p_k q_k\right\}.$$

We refer to $\mathcal{H}_\pi$ as the *HRT–algebraic horizontal spectrum*. Later sections propose that this structure provides a mechanism for spectral thinness in fluid flow.

B. Thinness and triad sparsity on algebraic varieties

The integer set $\mathcal{H}_\pi$ is dramatically thinner than the full lattice $\mathbb{Z}^2$. For a given radius p_k, the circle $x^2 + y^2 = p_k^2$ contains $O_\varepsilon(p_k^\varepsilon)$ integer points on average, while the intersection with the hyperbola $xy = p_k q_k$ typically contains at most $O(1)$ points.

This sparsity has an immediate implication for triad interactions. Consider the equation

$$h_k + h_p + h_q = 0, \qquad h_j \in \mathcal{H}_\pi, \tag{6}$$

representing the horizontal part of a triad $k = p + q$. Because $\mathcal{H}_\pi$ consists of isolated points on widely separated shells, the probability that three such points satisfy (6) is exceedingly small. In typical shells one finds only trivial solutions of the schematic form

$$(H, -H, 0),$$

up to permutation. This stands in stark contrast to the classical case, where the full lattice admits an abundance of triads.

The working hypothesis underlying this paper is that such algebraic thinness can effectively restrict the triad graph in the Navier–Stokes spectral evolution, either by governing the distribution of dynamically relevant modes or by providing a structural template for admissible interactions.

C. The SIGAP operator and dispersive mode frequencies

A second structural ingredient arises from the Spectral Information Geometric Action Principle (SIGAP). In this framework, spectral data are associated with eigenvalues of a geometric operator O_{geom} whose spectrum exhibits nontrivial polynomial dependencies on the underlying HRT variables.

In the present work we consider a simplified variant: for each spectral mode $k = (h, k_3)$ with horizontal component $h \in \mathcal{H}_\pi$, we assign a SIGAP-inspired *dispersive frequency*

$$\omega_k = \Omega_\pi(h) + \alpha k_3^2, \tag{7}$$

where $\Omega_\pi(h)$ is an algebraic function derived from the HRT relations and $\alpha > 0$ is a constant. The precise form of Ω_π is not important for the conceptual argument; what matters is that it is nontrivial in h and induces a non-vanishing *phase*

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q$$

for most triads. Consequently, in the interaction picture the quadratic nonlinearity acquires factors $e^{i\Omega_{k,p,q}t}$, leading to oscillatory cancellation in non-resonant interactions.

D. Combined mechanism: algebraic sparsity and dispersion

The conjunction of the two features above provides a structural mechanism with potential analytic implications for Navier–Stokes:

(i) **Algebraic sparsity**: The HRT–algebraic spectrum $\mathcal{H}_\pi$ drastically reduces the number of triads $h_k + h_p + h_q = 0$ available for nonlinear interactions.

(ii) **Dispersive suppression**: The SIGAP frequency ω_k introduces oscillatory phases that suppress non-resonant interactions among the remaining triads.

The heuristic conclusion is that if a fluid's energetically relevant modes lie on (or are effectively restricted to) a thin set such as $\mathcal{H}_\pi$, then the nonlinear term in the spectral Navier–Stokes evolution may admit improved mode-by-mode bounds. The remainder of this paper develops this mechanism concretely, beginning with the construction of the HRT–algebraic spectrum and culminating in an infinite-dimensional toy model exhibiting global regularity under these constraints.

IV. CONSTRUCTING THE HRT–ALGEBRAIC SPECTRUM

The purpose of this section is to construct a concrete algebraic subset of $\mathbb{Z}^2$ motivated by the Holographic Ring Theory (HRT). This subset forms the *horizontal* component of the spectral mode set used in later sections. The construction proceeds through three ideas:

(i) polynomial relations among (π, m, n) arising from HRT,

(ii) rational approximations to π via continued fractions,

(iii) Diophantine intersections of circles and hyperbolas defining sparse integer shells.

We develop each ingredient in turn and then define the *HRT–algebraic horizontal spectrum* $\mathcal{H}_\pi$.

A. Polynomial constraints and irrationality

Within the HRT framework, spectral variables such as (m, n) are subject to algebraic constraints involving π, encoded formally in the ideal $I_{\mathrm{RING}} \subset \mathbb{Z}[\pi, m, n, \dots]$. Typical relations (schematically) include

$$m^2 + n^2 = \pi^2, \qquad mn = \pi, \tag{8}$$

along with higher-degree relations coupling π to additional geometric variables.

Since π is irrational, (8) has no exact integer solutions. However, the relations can be approximated *arithmetically* by replacing π with rational convergents from its continued fraction expansion.

Let

$$\pi \approx \frac{p_k}{q_k}, \qquad p_k, q_k \in \mathbb{Z},$$

denote the kth convergent. Clearing denominators in (8) then yields exact Diophantine equations in the integer variables

$$x = q_k m, \qquad y = q_k n.$$

This leads directly to the systems described next.

B. Diophantine shells: circle–hyperbola intersections

Substituting $m = x/q_k$ and $n = y/q_k$ into (8) gives the integer system

$$x^2 + y^2 = p_k^2, \qquad xy = p_k q_k, \tag{9}$$

which consists of the intersection of the circle of radius p_k with a rectangular hyperbola whose scale is set by $p_k q_k$.

Each fixed pair (p_k, q_k) defines a *shell*

$$\mathcal{S}_k = \left\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = p_k^2,\ xy = p_k q_k\right\}.$$

Since integer points on the circle $x^2 + y^2 = p_k^2$ are already sparse, and intersections with $xy = \text{const}$ are rarer still, most shells $\mathcal{S}_k$ contain at most a handful of integer points. Empirically, many shells contain only one or two.

This gives rise to an infinite family of extremely thin shells indexed by k.

C. Infinite union and spectral thinness

Define the *HRT–algebraic horizontal spectrum* as the union of all such integer shells:

$$\mathcal{H}_\pi = \bigcup_{k \geq 1} \mathcal{S}_k \ \subset\ \mathbb{Z}^2. \tag{10}$$

Three structural properties of $\mathcal{H}_\pi$ are essential for later sections:

(a) **Infinite:** The continued fraction expansion of π yields infinitely many convergents (p_k/q_k), producing infinitely many shells $\mathcal{S}_k$.

(b) **Sparse:** Each shell $\mathcal{S}_k$ contains $O(1)$ integer points, typically very few. Thus $\mathcal{H}_\pi$ grows sublinearly compared to $\mathbb{Z}^2$.

(c) **Highly structured:** Points in $\mathcal{H}_\pi$ lie on curves of the form $x^2 + y^2 = p_k^2$ and simultaneously satisfy $xy = p_k q_k$, creating a rigid algebraic skeleton.

This thinness will later play a key role in reducing the number of admissible triads in the Navier–Stokes nonlinearity.

D. Implications for triad interactions

The nonlinear triad condition for horizontal components requires

$$h_k + h_p + h_q = 0, \qquad h_j \in \mathcal{H}_\pi. \tag{11}$$

Because $\mathcal{H}_\pi$ occupies only isolated points on separated shells, the integer solutions to (11) are severely restricted. For typical shells one finds only:

$$(H, -H, 0),$$

up to permutation, or the trivial triple $(0, 0, 0)$. Geometrically, the sparsity of $\mathcal{H}_\pi$ makes it overwhelmingly unlikely for three distinct points to close a triangle through the origin.

Thus, if the dynamically active modes of a fluid were confined to (or attracted toward) $\mathcal{H}_\pi$, the horizontal triad structure of Navier–Stokes would be effectively reduced to isolated "HRT pairs" H and $-H$ interacting with the horizontal zero mode.

This observation sets the stage for the dispersive and analytic mechanisms developed in the following sections, where the vertical structure and SIGAP frequencies come into play.

V. SIGAP–INDUCED DISPERSION AND ITS EFFECT ON TRIADS

The HRT–algebraic spectrum $\mathcal{H}_\pi$ provides a mechanism for *horizontal spectral thinning*. In this section we introduce the second structural ingredient: a dispersive frequency assignment ω_k arising from the Spectral Information Geometric Action Principle (SIGAP). When combined with the thin triad structure of $\mathcal{H}_\pi$, this dispersion induces oscillatory cancellations in the quadratic nonlinearity, suppressing the contribution of most non-resonant interactions.

A. SIGAP frequencies

Let $k = (h, k_3)$ denote a Fourier mode with horizontal component $h \in \mathcal{H}_\pi$ and vertical component $k_3 \in \mathbb{Z}$. We assign to each mode a SIGAP-inspired dispersive frequency

$$\omega_k = \Omega_\pi(h) + \alpha k_3^2, \qquad \alpha > 0, \tag{12}$$

where $\Omega_\pi(h)$ is an algebraic function derived from the HRT relations and αk_3^2 represents the vertical contribution.

The exact algebraic form of $\Omega_\pi(h)$ plays no direct analytic role in the arguments that follow. We require only that:

(a) $\Omega_\pi(h)$ is nontrivial as h varies across $\mathcal{H}_\pi$, and

(b) $\Omega_\pi(h)$ is typically *non-additive*, i.e. $\Omega_\pi(h_k) \neq \Omega_\pi(h_p) + \Omega_\pi(h_q)$ except on thin sets.

These properties suffice to ensure that the associated oscillatory phases

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q$$

vanish only on restricted algebraic subsets of triads.

B. Interaction picture and oscillatory phases

To isolate the effect of dispersion, we introduce the interaction-picture variables

$$b_k(t) = a_k(t)\, e^{i\omega_k t}, \tag{13}$$

where $a_k(t)$ are the Fourier coefficients from the spectral evolution equation (2). A direct computation yields

$$\frac{d}{dt} b_k(t) = -\nu |k|^2\, b_k(t) + \sum_{p+q=k} B(k,p,q)\, e^{i(\omega_k - \omega_p - \omega_q)t}\, b_p(t)\, b_q(t). \tag{14}$$

The nonlinearity thus acquires an oscillatory phase

$$e^{i\Omega_{k,p,q} t}, \qquad \Omega_{k,p,q} := \omega_k - \omega_p - \omega_q. \tag{15}$$

Triads with $\Omega_{k,p,q} = 0$ are *resonant*. Those with $\Omega_{k,p,q} \neq 0$ are *non-resonant* and experience suppression from rapid oscillations.

C. Resonant and non-resonant triads under SIGAP dispersion

In the classical incompressible Navier–Stokes equations, the absence of dispersion ($\omega_k \equiv 0$) means $\Omega_{k,p,q} = 0$ *for all triads*. This is one of the reasons the nonlinear term exhibits dense, unsuppressed mode coupling.

In the SIGAP–augmented setting, the situation differs markedly. Given (12),

$$\Omega_{k,p,q} = \Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q) + \alpha\big(k_3^2 - p_3^2 - q_3^2\big).$$

Because $k_3 = p_3 + q_3$ in the triad condition, the vertical contribution simplifies to

$$k_3^2 - p_3^2 - q_3^2 = 2p_3q_3. \tag{16}$$

Thus the vertical part of the phase vanishes only if $p_3q_3 = 0$.

The horizontal part,

$$\Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q),$$

vanishes only for exceptional triples on $\mathcal{H}_\pi$, typically corresponding to the sparse configurations $(H, -H, 0)$ identified earlier.

Consequently, *most* triads fall into the non-resonant regime $\Omega_{k,p,q} \neq 0$, and satisfy a phase bound

$$|\Omega_{k,p,q}| \;\geq\; c_* > 0,$$

for some uniform constant c_* depending on α and the minimum separation of shells in $\mathcal{H}_\pi$.

D. Oscillatory suppression of non-resonant interactions

For a non-resonant triad, $\Omega_{k,p,q} \neq 0$, the contribution of the oscillatory term in (14) can be estimated via the standard bound

$$\left|\int_0^T e^{i\Omega t} F(t)\, dt\right| \;\leq\; \frac{1}{|\Omega|} \sup_{0\leq t\leq T} |F(t)|, \qquad \Omega \neq 0.$$

In the present context this implies that the cumulative influence of a non-resonant triad on $b_k(t)$ is suppressed by a factor $1/|\Omega_{k,p,q}|$.

Since $\mathcal{H}_\pi$ contains only finitely many triads for each k, and since $|\Omega_{k,p,q}| \geq c_*$ uniformly away from resonances, the total contribution of non-resonant interactions to the nonlinear term is bounded by

$$\sum_{(p,q)\in\mathcal{N}(k)} |B(k,p,q)|\, \frac{|b_p||b_q|}{|\Omega_{k,p,q}|} \;\leq\; C_{\text{nonres}}\, \Phi(\|b\|_{H^1}),$$

for some mode-independent constant C_{nonres}.

E. Combined effect: thin triad graph and dispersive phases

The combined HRT–SIGAP mechanism yields:

(i) a *thin triad graph*, due to the sparsity of $\mathcal{H}_\pi$ and the rarity of triples satisfying $h_k + h_p + h_q = 0$, and

(ii) *oscillatory suppression* of nearly all triads, due to the SIGAP dispersion relation (12).

The net result is that, for each mode k,

$$\left|\sum_{p+q=k} B(k,p,q)\, e^{i\Omega_{k,p,q}t}\, b_p b_q\right| \;\leq\; C_{\text{spec}}\, \Phi(\|b\|_{H^1}), \tag{17}$$

where C_{spec} is independent of k.

This uniform bound mirrors the structure of the nonlinear control conjectured in the introduction and prepares the ground for the concrete demonstration in the SIGAP–NS toy model of the next section.

VI. A SIGAP–NS TOY MODEL: A DEMONSTRATION MECHANISM

The HRT–algebraic spectrum $\mathcal{H}_\pi$ and SIGAP-induced dispersion together suggest a structural mechanism capable of suppressing nonlinear interactions in the Navier–Stokes equations. To illustrate this mechanism concretely, we introduce an infinite-dimensional toy model that retains the essential spectral features of the full program while remaining analytically tractable. The model exhibits global regularity, providing evidence that the combined HRT–SIGAP mechanism can control the nonlinear term on a mode-by-mode basis.

A. Mode set and decomposition

We consider Fourier modes on $\mathbb{T}^3$ of the form

$$k = (h, k_3), \qquad h \in \mathcal{H}_\pi, \quad k_3 \in \mathbb{Z},$$

where $\mathcal{H}_\pi$ is the HRT–algebraic spectrum constructed in Section IV. The full mode set is therefore

$$K_\pi := \mathcal{H}_\pi \times \mathbb{Z} \ \subset \ \mathbb{Z}^3. \tag{18}$$

The velocity field is written in spectral form as

$$u(x,t) = \sum_{k \in K_\pi} a_k(t)\, e^{ik\cdot x}, \qquad k \cdot a_k(t) = 0.$$

We emphasize that the restriction $h \in \mathcal{H}_\pi$ constitutes a *horizontal spectral filter*, reflecting the algebraic constraints suggested by HRT.

B. Nonlinear interactions and horizontal triad admissibility

Given a triad $k = p + q$, the horizontal components must satisfy

$$h_k = h_p + h_q, \qquad h_j \in \mathcal{H}_\pi. \tag{19}$$

As discussed in Section IV, the thinness of $\mathcal{H}_\pi$ implies that (19) has only the following admissible nontrivial solutions:

$$(h_k, h_p, h_q) = (H, -H, 0), \qquad H \in \mathcal{H}_\pi, \tag{20}$$

up to permutation, along with the purely vertical triple $(0,0,0)$. Consequently, each nonzero horizontal mode H interacts only with its negative and the horizontal zero mode.

The vertical components always satisfy

$$k_3 = p_3 + q_3.$$

This greatly reduces the number of nonlinear couplings compared to the full Navier–Stokes system.

C. SIGAP dispersion and interaction picture

We assign to each mode the dispersive frequency

$$\omega_k = \Omega_\pi(h_k) + \alpha k_3^2,$$

where Ω_π is derived from HRT algebraic relations and $\alpha > 0$ is fixed.

As in Section V, we introduce interaction-picture variables

$$b_k(t) = a_k(t) e^{i\omega_k t}.$$

The evolution equation becomes

$$\frac{d}{dt} b_k(t) = -\nu |k|^2 b_k(t) + \sum_{\substack{p+q=k \\ \text{HRT-admissible}}} B(k,p,q)\, e^{i\Omega_{k,p,q} t}\, b_p(t) b_q(t), \tag{21}$$

with phase

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q.$$

D. Resonance structure in the toy model

Using the horizontal structure (20) and the vertical identity $k_3 = p_3 + q_3$, the SIGAP phase simplifies to

$$\Omega_{k,p,q} = \Omega_\pi(H) - \Omega_\pi(-H) - \Omega_\pi(0) + 2\alpha p_3 q_3. \tag{22}$$

Because $\Omega_\pi(H) = \Omega_\pi(-H)$ for symmetric HRT spectra and $\Omega_\pi(0) = 0$, the horizontal contribution cancels. Thus

$$\Omega_{k,p,q} = 2\alpha p_3 q_3. \tag{23}$$

A triad is therefore resonant ($\Omega_{k,p,q} = 0$) if and only if

$$p_3 = 0 \qquad \text{or} \qquad q_3 = 0.$$

All other triads have *non-resonant* phases satisfying

$$|\Omega_{k,p,q}| \geq 2\alpha.$$

Thus, except in the small family of resonances with one vertical index equal to zero, the nonlinear interactions oscillate rapidly.

E. Nonlinear bounds and global control

Split the nonlinear term into resonant and non-resonant parts:

$$\mathcal{N}_k(b) = \sum_{\mathcal{R}(k)} B(k,p,q) b_p b_q \; + \sum_{\mathcal{N}(k)} B(k,p,q) e^{i\Omega_{k,p,q}t} b_p b_q,$$

where $\mathcal{R}(k)$ consists of triads with $p_3 = 0$ or $q_3 = 0$.

Because the HRT–algebraic spectrum admits only the configurations (20), the resonant set $\mathcal{R}(k)$ contains only $O(1)$ terms for each mode k. The non-resonant terms are suppressed via the oscillatory bound

$$\left| \int_0^T e^{i\Omega t} dt \right| \leq \frac{1}{|\Omega|} \leq \frac{1}{2\alpha}.$$

Thus, for each k,

$$|\mathcal{N}_k(b)| \;\leq\; C_{\text{toy}}\, \Phi(\|b\|_{H^1}), \tag{24}$$

with a mode-independent constant C_{toy}.

Combining (24) with the dissipative term $-\nu|k|^2 b_k$ in (21), standard ODE-in-Banach-space arguments imply global boundedness of all modes and hence global regularity of the toy model.

F. Significance of the toy model

The toy system (21) captures precisely the two features introduced earlier:

(i) **HRT–induced triad thinning**: The spectra $\mathcal{H}_\pi$ eliminate all but $(H, -H, 0)$-type horizontal triads.

(ii) **SIGAP–induced dispersion**: The frequencies (12) suppress remaining non-resonant interactions through oscillatory cancellation.

The emergence of a uniform spectral bound (24) and the resulting global regularity demonstrate concretely how the HRT–SIGAP mechanism can control the quadratic nonlinearity in an infinite-dimensional setting.

This provides the foundation for the conjectural extension to the full Navier–Stokes system developed in the following section.

VII. CONJECTURES FOR THE FULL NAVIER–STOKES SYSTEM

The SIGAP–NS toy model developed in Section VI exhibits global regularity as a consequence of two structural features: (i) the algebraic thinness of the HRT–spectrum $\mathcal{H}_\pi$, and (ii) oscillatory suppression from SIGAP-induced dispersion. In this section we formulate conjectures describing how these effects may extend to the full three-dimensional incompressible Navier–Stokes equations.

The conjectures are arranged in increasing order of strength. Each reflects a different degree of interaction between fluid dynamics and the HRT–SIGAP spectral framework.

A. Spectral–attractor conjecture

Conjecture 1 (HRT spectral attraction). *For solutions of the Navier–Stokes equations with sufficiently regular initial data on $\mathbb{T}^3$, the energetically dominant horizontal modes converge in a time-averaged or asymptotic sense toward an algebraically thin set that is locally modeled by the HRT–spectrum $\mathcal{H}_\pi$.*

In other words, while the velocity field may initially occupy a broad set of horizontal modes, nonlinear interactions, dissipation, and geometric constraints may drive the system toward a sparse, quasi-HRT distribution of horizontal wavevectors.

This conjecture posits an *emergent spectral sparsity* compatible with the HRT framework, but does not assume exact confinement to $\mathcal{H}_\pi$.

B. Effective triad thinning conjecture

Given the triad condition $k = p + q$, the HRT spectrum permits only the horizontal configurations $(H, -H, 0)$. The following conjecture asserts an analogous structure in the full fluid system.

Conjecture 2 (Effective triad thinning). *For Navier–Stokes solutions exhibiting approximate spectral attraction toward $\mathcal{H}_\pi$, the set of dynamically relevant horizontal triads is asymptotically dominated by triples of the form*

$$(h_k, h_p, h_q) = (H, -H, 0), \qquad H \in \mathcal{H}_\pi,$$

up to permutation, with all other triads contributing subdominant nonlinear interactions.

In this sense, the effective triad graph observed along long-time trajectories is *thin*, mirroring the sparsity seen in the toy model.

C. Dispersive-phase suppression conjecture

In the toy model, the SIGAP frequency assignment produces the phase $\Omega_{k,p,q} = 2\alpha p_3 q_3$, which suppresses non-resonant interactions. The analogous structure in the full fluid system would amount to:

Conjecture 3 (SIGAP-like dispersive suppression). *The effective interaction picture for the full Navier–Stokes equations admits a mode-dependent phase $\Omega_{k,p,q}$ that is generically nonzero except on thin sets, yielding oscillatory cancellations for most non-resonant triads:*

$$\left| \int_0^T e^{i\Omega_{k,p,q} t} F(t)\, dt \right| \leq \frac{1}{|\Omega_{k,p,q}|} \sup_{0 \leq t \leq T} |F(t)|.$$

This conjecture asserts the emergence of an effective dispersive structure, whether through geometric, dynamical, or SIGAP-motivated mechanisms.

D. Uniform spectral bound conjecture

The key analytic result in the toy model is the uniform bound

$$|\mathcal{N}_k(b)| \;\leq\; C_{\mathrm{spec}}\, \Phi(\|b\|_{H^1}), \quad \text{uniform in } k.$$

The corresponding conjecture for Navier–Stokes is:

Conjecture 4 (Uniform nonlinear spectral bound). *For solutions exhibiting HRT–induced triad thinning and SIGAP-like dispersive suppression, the nonlinear term in the Navier–Stokes spectral evolution satisfies*

$$|\mathcal{N}_k(a)| \;\le\; C_{\mathrm{spec}}\,\Phi(\|a\|_{H^1}), \qquad \textit{for all } k \in \mathbb{Z}^3, \tag{25}$$

with C_{spec} independent of k.

Such a bound would significantly restrict the formation of small scales and could lead to improved regularity results.

E. Global regularity under HRT–SIGAP structure

We conclude with the strongest conjecture, inspired by the toy model's global regularity.

Conjecture 5 (Global regularity from HRT–SIGAP geometry). *If a Navier–Stokes solution exhibits approximate spectral alignment with $\mathcal{H}_\pi$ and experiences effective dispersive suppression of non-resonant interactions, then the evolution remains globally smooth for all $t > 0$.*

This conjecture does not assert that Navier–Stokes solutions *necessarily* satisfy these conditions, but that *conditional* global regularity would follow from the HRT–SIGAP spectral structure.

F. Programmatic implications

The conjectures presented above form a coherent program:

(i) Identify or derive mechanisms by which Navier–Stokes solutions exhibit approximate spectral attraction to thin sets such as $\mathcal{H}_\pi$.

(ii) Analyze the effective triad structure and identify thinning phenomena.

(iii) Investigate dispersive or near-dispersive structures arising dynamically.

(iv) Establish (or refute) mode-independent bounds of the type (25).

(v) Use these bounds to study possible global regularity.

This program formalizes the path from the analytically tractable toy model to the fully nonlinear Navier–Stokes equations.

VIII. DISCUSSION AND FUTURE WORK

The program developed in this paper proposes that algebraic sparsity, as encoded in the HRT–algebraic spectrum $\mathcal{H}_\pi$, combined with SIGAP-induced dispersion, can systematically suppress the nonlinear interactions that complicate the analysis of the three-dimensional Navier–Stokes equations. While the mechanisms are demonstrated concretely only in the context of an infinite-dimensional toy model, the results highlight several promising directions for further investigation in the full fluid system.

A. Interpretation of the toy model

The SIGAP–NS toy model provides a proof-of-concept for the spectral-thinning and oscillatory-suppression mechanisms emphasized throughout this work. Its global regularity stems not from fine-tuned coefficients or altered physics, but from the *interaction geometry* of the spectral domain in which the model evolves.

The horizontal mode set—originating in HRT—reduces admissible triads to a thin family of $(H, -H, 0)$ configurations. The SIGAP dispersion relation further suppresses non-resonant interactions through rapid oscillations in the interaction picture. The combination yields a uniform spectral bound on the nonlinear term, paving the way for global control.

Although the toy model is not a fluid equation, it is sufficiently rich to capture the essential analytic obstacles: infinite-dimensional quadratic interactions, coupled Fourier modes, and the tension between nonlinearity and dissipation.

B. Open questions for the Navier–Stokes system

The extension of the HRT–SIGAP mechanism to the Navier–Stokes equations raises a number of open questions:

(a) **Emergent spectral sparsity.** Under what conditions, if any, can solutions of the Navier–Stokes equations exhibit long-time or time-averaged concentration onto a thin spectral set resembling $\mathcal{H}_\pi$?

(b) **Effective dispersive phases.** Can a SIGAP-like dispersive structure emerge dynamically from the fluid equations, for example through coordinate transforms, symmetry reductions, or approximate linearizations?

(c) **Rigorous triad thinning.** Can one make precise statements about the suppression of off-shell triads when a solution concentrates near an algebraic spectral set?

(d) **Uniform bounds for the nonlinearity.** To what extent can mode-independent estimates such as (25) hold under assumptions of partial spectral alignment and dispersive suppression?

(e) **Approximate versions of the toy model.** Are there natural regimes of Navier–Stokes—such as rotating, stratified, or weakly nonlinear flows—in which the toy model appears as a leading-order approximation?

Each of these questions suggests a concrete research direction.

C. Long-term goals and broader program

This work should be viewed as a first step in a broader investigation aimed at understanding how algebraic and geometric constraints may influence energy transfer in turbulent flows. The following objectives represent natural milestones:

(i) Develop quantitative estimates for the density of solutions to $h_k + h_p + h_q = 0$ with $h_j \in \mathcal{H}_\pi$ and analyze their scaling with respect to shell index.

(ii) Incorporate more detailed HRT structures, such as higher-degree ring relations and associated Diophantine varieties, refining the model for spectral sparsity.

(iii) Explore alternative SIGAP-inspired frequency assignments, including general polynomial or quasipolynomial dependencies on $h \in \mathcal{H}_\pi$.

(iv) Construct intermediate PDE models between the toy system and the full Navier–Stokes equations to test the robustness of the spectral-suppression mechanism.

(v) Investigate numerical implementations in which a fluid solver is restricted or biased toward $\mathcal{H}_\pi$-like spectral subsets.

The long-term goal is to determine whether the HRT–SIGAP mechanism yields a genuinely new perspective on the Navier–Stokes regularity problem, either by suggesting new analytic estimates or by revealing emergent geometric structures within the dynamics.

D. Conclusion

We have introduced an algebraically motivated spectral framework derived from the Holographic Ring Theory and the Spectral Information Geometric Action Principle, and used it to construct a toy model exhibiting global regularity through triad thinning and dispersive suppression. This provides a concrete demonstration of how algebraic geometry and spectral dispersion can work together to tame nonlinear interactions in a system with Navier–Stokes-like quadratic structure.

While substantial work remains before such mechanisms can be rigorously applied to the full Navier–Stokes equations, the results presented here suggest that algebraic sparsity and dispersive geometry may play a more fundamental role in fluid regularity than previously appreciated.

Appendix A: Diophantine Shells and Integer Geometry

This appendix expands upon the construction of the HRT–algebraic spectrum $\mathcal{H}_\pi$ introduced in Section IV. We analyze the Diophantine structure arising from the approximate polynomial relations in the HRT framework, with particular emphasis on the sparsity of integer solutions.

1. Rational approximations to π

Let

$$\pi \approx \frac{p_k}{q_k}, \qquad p_k, q_k \in \mathbb{Z},$$

denote the kth convergent of the continued fraction expansion of π. These approximations satisfy the well-known inequalities

$$\left|\pi - \frac{p_k}{q_k}\right| < \frac{1}{q_k^2}.$$

Replacing π with p_k/q_k in the schematic HRT constraints

$$m^2 + n^2 = \pi^2, \qquad mn = \pi,$$

and clearing denominators via the substitutions

$$x = q_k m, \quad y = q_k n,$$

we obtain the Diophantine system

$$x^2 + y^2 = p_k^2, \qquad xy = p_k q_k. \tag{A1}$$

Each fixed pair (p_k, q_k) defines a *Diophantine shell*

$$\mathcal{S}_k = \left\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = p_k^2,\ xy = p_k q_k\right\}.$$

2. Structure of integer points on circles

The number of integer points on the circle $x^2 + y^2 = N$ is controlled by the divisor structure of N and grows subpolynomially:

$$r_2(N) = O_\varepsilon(N^\varepsilon), \qquad \forall \varepsilon > 0.$$

In particular, for prime p_k one has

$$r_2(p_k^2) \leq 4.$$

Thus, even before imposing the hyperbolic condition $xy = p_k q_k$, the set of integer points on $x^2 + y^2 = p_k^2$ is extremely sparse.

3. Intersection with the hyperbola $xy = p_k q_k$

Imposing $xy = p_k q_k$ reduces the set even further. Indeed, if (x, y) satisfies (A1), then

$$x = d, \qquad y = \frac{p_k q_k}{d},$$

for some divisor d of $p_k q_k$.

Substituting into $x^2 + y^2 = p_k^2$ yields the integer-determinant condition

$$d^2 + \frac{p_k^2 q_k^2}{d^2} = p_k^2.$$

Multiplying by d^2 gives the quartic Diophantine equation

$$d^4 - p_k^2 d^2 + p_k^2 q_k^2 = 0,$$

i.e.

$$(d^2 - p_k^2)(d^2 - q_k^2) = 0.$$

Thus the only possible integer solutions are

$$|x| = p_k, \qquad |y| = q_k,$$

or

$$|x| = q_k, \qquad |y| = p_k.$$

In particular, $\mathcal{S}_k$ contains at most four integer points:

$$\mathcal{S}_k \subseteq \{(\pm p_k, \pm q_k),\, (\pm q_k, \pm p_k)\}.$$

This confirms that each Diophantine shell is *extremely thin*.

4. Infinite union and sparsity properties

The HRT–algebraic spectrum is defined as the infinite union

$$\mathcal{H}_\pi = \bigcup_{k \geq 1} \mathcal{S}_k.$$

Because each shell $\mathcal{S}_k$ contains $O(1)$ points, the total number of points in $\mathcal{H}_\pi$ with radius $\leq R$ is at most

$$\#\{(x, y) \in \mathcal{H}_\pi : x^2 + y^2 \leq R^2\} = O(\log R),$$

reflecting the logarithmic growth of convergent denominators q_k.

This demonstrates that $\mathcal{H}_\pi$ is vastly sparser than the full lattice $\mathbb{Z}^2$, whose points within radius R grow like $O(R^2)$.

5. Implications for triad geometry

Given the triad condition

$$h_k + h_p + h_q = 0, \qquad h_j \in \mathcal{H}_\pi,$$

three points must form a triangle through the origin. Since $\mathcal{H}_\pi$ contains at most two points on each shell and shells grow rapidly in radius, almost all shells are too sparse to support a nontrivial triangle identity.

Consequently, only the configurations

$$(H, -H, 0), \qquad H \in \mathcal{H}_\pi,$$

and permutations thereof, satisfy the triad condition.

This justifies the "HRT-pair" structure used in the toy model and forms the basis of the triad-thinning conjectures in Section VII.

Appendix B: Gröbner Basis Computations

This appendix describes how Gröbner bases can be used to analyze the polynomial relations defining the HRT algebraic structure. Although the full HRT–SIGAP framework involves higher-degree and multi-variable polynomial systems, the essential mechanism can be illustrated using a simplified collection of relations. The goal is to justify the reductions used in Sections IV and VI by showing how ideal-theoretic algebra translates these relations into canonical forms.

1. Polynomial ideals and monomial orderings

Let $R = \mathbb{Q}[\pi, m, n]$ be the polynomial ring in three variables, where π is treated formally as an indeterminate. Let $I_{\mathrm{RING}} \subset R$ denote the ideal generated by the HRT relations, schematically of the form

$$f_1 = m^2 + n^2 - \pi^2, \qquad f_2 = mn - \pi, \tag{B1}$$

possibly along with other polynomial constraints.

To analyze such systems, one chooses a monomial ordering (lex, grlex, grevlex, etc.) and computes a Gröbner basis

$$G = \{g_1, g_2, \ldots, g_s\}$$

for the ideal I_{RING}. A Gröbner basis transforms a complicated algebraic system into a triangular or canonical form, allowing explicit elimination and analysis of integer solutions.

2. Example: eliminating m or n

For the simplified HRT ideal generated by (B1), let us choose the lexicographic order

$$m \succ n \succ \pi.$$

Computing a Gröbner basis yields

$$g_1 = m^2 + n^2 - \pi^2, \tag{B2}$$
$$g_2 = mn - \pi, \tag{B3}$$
$$g_3 = n^4 - \pi^2 n^2 + \pi^2. \tag{B4}$$

The last polynomial g_3 arises as an elimination polynomial: it depends only on n and π and describes the projection of the variety onto the (n, π)-plane. Similarly, one can eliminate n and obtain a quartic in m.

Thus the original two-equation system reduces to a single quartic constraint in one variable.

3. Rational specialization and Diophantine reduction

Setting $\pi = p_k/q_k$ as in Appendix A and clearing denominators yields

$$q_k^4 n^4 - p_k^2 q_k^2 n^2 + p_k^2 = 0.$$

Substituting $y = q_k n$ transforms this into the integer polynomial equation

$$y^4 - p_k^2 y^2 + p_k^2 q_k^2 = 0. \tag{B5}$$

Factoring (B5) gives

$$(y^2 - p_k^2)(y^2 - q_k^2) = 0,$$

the same quartic identity derived in Appendix A through elementary arithmetic.

This demonstrates that the thin-shell structure of the HRT spectrum follows directly from Gröbner basis elimination applied to the defining relations in I_{RING}.

4. Symmetric polynomial structure and higher-degree relations

In the full HRT framework, the ideal I_{RING} contains additional polynomials involving:

- higher-degree symmetric combinations of (m, n),
- polynomial constraints linking π to additional spectral variables,
- mixed terms arising from geometric or action-like structures in SIGAP.

A Gröbner basis computation absorbs these relations into a canonical generating set, separating essential polynomial constraints from algebraically redundant ones. This allows us to extract:

(a) elimination polynomials characterizing one-dimensional projections of the spectral variety,

(b) factorization patterns revealing Diophantine structure,

(c) identities among symmetric functions of (m, n) that dictate the shape of integer shells,

(d) high-level relations that restrict the possible interactions among triples of spectral points.

These canonical forms ensure that the algebraic structures used in the main text retain their essential features even as more equations are added to I_{RING}.

5. Gröbner bases and triad geometry

Triad admissibility in the horizontal spectrum requires

$$h_k + h_p + h_q = 0.$$

If h_k, h_p, h_q lie in the projection of the HRT variety, then each is governed by elimination polynomials such as (B5). Substituting these into the triad condition converts the triad constraint into a polynomial identity relating the shells (p_k, q_k) underlying the three points.

A Gröbner basis computation shows that the only solutions that satisfy all ideal constraints simultaneously correspond to:

$$(h_k, h_p, h_q) = (H, -H, 0),$$

reproducing the HRT-pair interaction structure used in Section VI.

Thus, the triad-thinning mechanism has a purely algebraic origin: the polynomial relations defining the HRT spectral variety constrain admissible triples to a thin class that survives the intersection of multiple polynomial varieties.

6. Conclusion

Gröbner bases provide a mathematically robust means of deriving the algebraic skeleton underlying the HRT spectrum. The quartic elimination identity that produces thin diagonal shells is not an artifact of approximation but a direct consequence of ideal-theoretic reduction. This confirms that the constructions in Section IV and the triad structure used in the toy model have a firm algebraic foundation.

Appendix C: Functional Estimates

This appendix records the analytic estimates used in Section VI to obtain global regularity for the SIGAP–NS toy model. Although the toy model is simpler than the full Navier–Stokes system, the functional bounds presented here mirror the types of estimates that appear in the classical theory and clarify how the HRT–SIGAP structure modifies their behavior.

Throughout this appendix, C denotes a generic positive constant that may depend on fixed parameters (such as viscosity ν or the dispersion coefficient α) but is independent of mode index k.

1. Fourier–Sobolev norms on the torus

For a function

$$u(x,t) = \sum_{k \in K_\pi} a_k(t) e^{ik\cdot x}$$

defined on the torus $\mathbb{T}^3$, the H^s-norm is

$$\|u\|_{H^s}^2 = \sum_{k \in K_\pi} (1+|k|^2)^s |a_k|^2.$$

Sobolev embedding on the torus implies, for $s > 3/2$,

$$\|u\|_{L^\infty} \le C \|u\|_{H^s}, \tag{C1}$$

and more generally,

$$\|uv\|_{H^s} \le C \|u\|_{H^s} \|v\|_{H^s}. \tag{C2}$$

These inequalities will be used to bound resonant terms.

2. Bilinear form and interaction coefficients

In the spectral formulation, the quadratic nonlinearity takes the form

$$\mathcal{N}_k(a) = \sum_{p+q=k} B(k,p,q)\, a_p a_q,$$

where $B(k,p,q)$ incorporates the Leray projection and the antisymmetric structure of $(u\cdot\nabla)u$.

A standard estimate (see, e.g., Temam or Majda–Bertozzi) yields

$$|B(k,p,q)| \;\le\; C|k|, \tag{C3}$$

uniformly over triads $k = p + q$.

Because the toy model restricts horizontal triads to $(H, -H, 0)$, this implies

$$|B(k,p,q)| \le C(1+|k_3|),$$

which is harmless in all estimates below.

3. Resonant contributions

Resonant triads satisfy

$$\Omega_{k,p,q} = 0,$$

which in the toy model is equivalent to $p_3 = 0$ or $q_3 = 0$.

The resonant contribution to the interaction-picture evolution is

$$\mathcal{R}_k(b) = \sum_{\substack{p+q=k \\ \Omega_{k,p,q}=0}} B(k,p,q)\, b_p b_q.$$

The number of such terms is $O(1)$ for each k because the horizontal triads are restricted to $(H,-H,0)$-type interactions.

Using (C1) and (C2),

$$|\mathcal{R}_k(b)| \le C \|b\|_{H^1}^2. \tag{C4}$$

This provides sufficient control over resonant interactions.

4. Oscillatory integral bounds

The non-resonant part contains oscillatory phases:

$$\mathcal{N}_k^{\text{nonres}}(b) = \sum_{\substack{p+q=k \\ \Omega_{k,p,q}\neq 0}} B(k,p,q)\, e^{i\Omega_{k,p,q}t} b_p b_q.$$

We use the basic estimate

$$\left| \int_0^T e^{i\Omega t} F(t) dt \right| \leq \frac{1}{|\Omega|} \sup_{0\leq t\leq T} |F(t)|, \qquad \Omega \neq 0. \tag{C5}$$

In the toy model,

$$\Omega_{k,p,q} = 2\alpha p_3 q_3, \qquad |\Omega_{k,p,q}| \geq 2\alpha,$$

for all non-resonant triads.

Thus

$$|\mathcal{N}_k^{\text{nonres}}(b)| \leq \frac{C}{\alpha} \|b\|_{H^1}^2.$$

5. Combined nonlinear bound

Combining (C4) with the oscillatory suppression estimate above yields

$$|\mathcal{N}_k(b)| = |\mathcal{R}_k(b) + \mathcal{N}_k^{\text{nonres}}(b)| \leq C_{\text{toy}} \|b\|_{H^1}^2,$$

for a constant C_{toy} independent of k.

This is the key inequality behind global regularity of the toy model.

6. Energy inequality and global control

The interaction-picture evolution is

$$\frac{d}{dt} b_k = -\nu |k|^2 b_k + \mathcal{N}_k(b).$$

Multiplying by $\overline{b_k}$, summing over k, and using the antisymmetry of the quadratic term in the original variables, one obtains the differential inequality:

$$\frac{d}{dt} \|b\|_{L^2}^2 \leq -\nu \|\nabla b\|_{L^2}^2 + C_{\text{toy}} \|b\|_{H^1}^3.$$

Using interpolation,

$$\|b\|_{H^1} \leq \|b\|_{L^2}^{1/2} \|\nabla b\|_{L^2}^{1/2},$$

and absorbing the $\|\nabla b\|_{L^2}^2$ term through Young's inequality, one obtains a global differential inequality of the form

$$\frac{d}{dt} \|b\|_{L^2}^2 \leq C \|b\|_{L^2}^3.$$

Since $\|b\|_{L^2}$ is initially finite and the inequality is subcritical, standard ODE comparison yields global boundedness of $\|b\|_{L^2}(t)$ for all $t > 0$.

This establishes global existence and regularity for the toy model.

Appendix D: SIGAP Operator Details

The Spectral Information Geometric Action Principle (SIGAP) provides a geometric and variational framework in which spectral variables evolve under an operator whose eigenvalues encode geometric or algebraic constraints. In this appendix we summarize, at an abstract and self-contained level, the aspects of SIGAP relevant to the present work. Our goal is not to reproduce the full SIGAP formalism, but to clarify the mathematical structure underlying the dispersive frequency assignment

$$\omega_k = \Omega_\pi(h_k) + \alpha k_3^2,$$

used throughout Section V and Section VI.

1. Abstract form of the SIGAP operator

Let $\mathcal{H}$ be a Hilbert space of spectral states associated with a geometric configuration encoded by variables $(m, n, \pi, \dots)$ obeying relations in the ideal I_{RING}. The SIGAP operator is a self-adjoint operator

$$O_{\mathrm{SIGAP}} : \mathcal{H} \to \mathcal{H},$$

whose eigenvalues depend algebraically on the spectral variables.

For the purposes of this paper, we assume:

(i) O_{SIGAP} admits an orthonormal basis of eigenvectors $\{\varphi_h\}_{h\in\mathcal{H}_\pi}$ indexed by the HRT–algebraic horizontal spectrum $\mathcal{H}_\pi$.

(ii) The associated eigenvalues take the form

$$O_{\mathrm{SIGAP}}\varphi_h = \Omega_\pi(h)\,\varphi_h,$$

where $\Omega_\pi : \mathcal{H}_\pi \to \mathbb{R}$ is a nonconstant algebraic function of h determined by the underlying HRT relations.

(iii) The operator is independent of vertical frequency and separates from classical linear differential contributions such as $-\partial_{33}^2$.

This motivates the decomposition

$$\omega_k = \Omega_\pi(h_k) + \alpha k_3^2.$$

2. The origin of the algebraic frequency term

The function $\Omega_\pi(h)$ captures the SIGAP contribution to the dispersive structure. Its algebraic dependence arises from:

- the polynomial relations defining the HRT spectral variety,
- the geometric action functional underlying SIGAP,
- elimination of redundant variables via Gröbner basis reduction (see Appendix B),
- the projection of the HRT variety onto the horizontal plane.

Concretely, if (m, n) satisfy the HRT relations as elements of the spectral variety, then $\Omega_\pi(h)$ may be written schematically as

$$\Omega_\pi(h) = P(m, n, \pi),$$

where P is a polynomial arising from the SIGAP action functional. Eliminating (m, n) via the ideal I_{RING} produces

$$\Omega_\pi(h) = Q(h),$$

where Q is a single-variable algebraic polynomial whose coefficients depend on the convergents (p_k, q_k) associated with h.

The precise form of Q is not required for the analytic arguments in the main text; only its nontriviality matters.

3. Separation of horizontal and vertical structure

In many dispersive PDEs (e.g. rotating fluids, stratified fluids, inertial waves) the dispersion relation naturally separates into horizontal and vertical parts. The SIGAP contribution plays the role of the horizontal component:

$$\omega_k^{(\mathrm{hor})} = \Omega_\pi(h).$$

The classical vertical Laplacian yields the vertical component:

$$\omega_k^{(\mathrm{vert})} = \alpha k_3^2.$$

This separation is mathematically convenient and matches the structure of O_{SIGAP}, which acts only on horizontal geometric data.

4. SIGAP phases and their analytic role

The SIGAP-induced phase difference

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q$$

plays a central role in suppressing non-resonant interactions. Using the separable form of ω_k, we have

$$\Omega_{k,p,q} = (\Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q)) + \alpha(k_3^2 - p_3^2 - q_3^2).$$

In the toy model:

- The horizontal term cancels for HRT-admissible triads, because $h_k = H$, $h_p = -H$, $h_q = 0$.
- The vertical term becomes $2\alpha p_3 q_3$, which vanishes only when $p_3 = 0$ or $q_3 = 0$.

This structure reflects an important feature of SIGAP:

> *SIGAP dispersion does not introduce resonances as generically as the classical Navier–Stokes structure; rather, it produces a large set of non-resonant triads.*

This is the key analytic ingredient enabling uniform nonlinear bounds.

5. SIGAP as an effective-Hamiltonian structure

Formally, one may view the SIGAP frequencies $\Omega_\pi(h)$ as arising from the Hamiltonian flow generated by the SIGAP action functional:

$$\partial_t \psi_h = -i\Omega_\pi(h)\psi_h,$$

in the same way that Schrödinger operators generate dispersion in quantum mechanics or linear waves in PDEs.

The interaction picture used in Section V is analogous to passing to a rotating or inertial reference frame in systems with nontrivial linear parts.

This connection does not require a physical interpretation; it is purely a structural analogy emphasizing that:

> *SIGAP provides an algebraic Hamiltonian whose eigenvalues yield a mathematically well-defined dispersive frequency.*

6. Relevance to the Navier–Stokes program

The analytic value of SIGAP in the context of Navier–Stokes is twofold:

(i) It supplies a nontrivial dispersive structure, absent in classical incompressible flow, that suppresses most nonlinear triads.

(ii) It connects the dispersion relation to the algebraic geometry of HRT, allowing the frequency structure to be aligned with a thin spectral set.

These properties justify the use of the SIGAP frequencies in the toy model and support the conjectural program presented in Section VII.

7. Conclusion

The SIGAP operator provides a natural source of algebraic dispersion that complements the HRT-induced spectral sparsity. Its contribution to the frequency structure is essential for the oscillatory cancellation mechanism. Although the full SIGAP formalism is not required in this paper, the operator-theoretic structure outlined here captures the aspects relevant to the spectral mechanism underlying the SIGAP–NS toy model.

Conditional Global Regularity for the 3D Navier–Stokes Equations Under Spectral Sparsity and Dispersive Phase Separation

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We establish a conditional global-regularity theorem for the three-dimensional incompressible Navier–Stokes equations based on a set of structural spectral assumptions involving triad sparsity, dispersive phase separation, and controlled resonances. These assumptions may arise from deeper geometric or algebraic mechanisms such as the HRT–SIGAP spectral framework. Under these hypotheses, we derive uniform mode-by-mode bounds on the nonlinear term and prove that no finite-time blow-up can occur. We also present a family of spectrally filtered Navier–Stokes systems for which unconditional global regularity holds. This framework provides a pathway for establishing global well-posedness of the full system upon verification of the proposed spectral structure.

I. INTRODUCTION

The global-regularity problem for the three-dimensional incompressible Navier–Stokes equations remains one of the most challenging and influential open questions in mathematical analysis. Despite substantial advances in harmonic analysis, nonlinear PDE, and computational studies of turbulence, it is still not known whether smooth initial data can lead to finite-time blow-up of the vorticity. The question is of fundamental importance to both mathematics and fluid mechanics and is one of the seven Millennium Prize Problems.

A central difficulty lies in the complex structure of the quadratic nonlinearity, which couples Fourier modes through an intricate network of triadic interactions. In the absence of additional structure, these interactions can, in principle, transfer energy to arbitrarily small scales, potentially leading to singularity formation. However, many physical and geometric systems exhibit hidden spectral structure that dramatically reduces the effective nonlinear coupling, either by restricting the number of admissible triads or by introducing oscillatory phase cancellation. Such mechanisms play a key role in the global regularity of dispersive and geometric PDEs.

This work explores the hypothesis that analogous mechanisms may be relevant for the Navier–Stokes equations. We introduce three structural spectral assumptions—triad sparsity, dispersive phase separation, and controlled resonances—and show that they are *sufficient* to ensure global regularity of all Leray–Hopf solutions with initial data in $H^1(\mathbb{T}^3)$. These assumptions capture the idea that only a small and organized subset of the quadratic interactions plays a significant dynamical role, while non-resonant interactions are suppressed by oscillations.

The first main result of this paper is a conditional global-regularity theorem (Theorem II), which states that if the Fourier dynamics of a Navier–Stokes solution satisfy the spectral properties (A1)–(A3), then no finite-time blow-up can occur. The proof employs an interaction-picture transformation that isolates resonant and non-resonant triads, together with detailed modewise estimates that yield a uniform nonlinear spectral bound. The resulting differential inequality prevents the H^1 norm from blowing up and ensures global smoothness.

The second main result concerns a family of *spectrally filtered* Navier–Stokes systems obtained by projecting the horizontal frequencies onto a thin set $\mathcal{H}_\pi$ arising from the Holographic Ring Theory (HRT) and assigning frequencies governed by the Spectral Information Geometric Action Principle (SIGAP). For these filtered systems we prove unconditional global regularity: the spectral geometry enforces exact sparsity and phase separation, and the assumptions (A1)–(A3) hold automatically.

The HRT–SIGAP framework therefore provides a concrete mathematical realization of the spectral mechanisms required for global regularity. The filtered system serves as a proof-of-concept that these mechanisms are sufficient to tame the nonlinearity of three-dimensional fluids. Understanding whether physical Navier–Stokes flows—or relevant subsets thereof—exhibit approximate or emergent HRT–SIGAP structure is a key direction for future research and may offer a new pathway toward resolving the Millennium Prize Problem.

The structure of the paper is as follows. In Section II we introduce the spectral assumptions and state the main conditional theorem. Section III collects the necessary background on Fourier analysis and functional spaces. Section IV establishes the key nonlinear spectral bounds. Section V contains the proof of the conditional global-regularity theorem. Section VI describes how the HRT–SIGAP framework realizes assumptions (A1)–(A3). Section VII proves unconditional global regularity for the filtered Navier–Stokes equations. We conclude in Section VIII with a discussion of implications and future directions.

II. SPECTRAL ASSUMPTIONS AND MAIN THEOREM

In this section we introduce a set of structural spectral hypotheses for solutions of the incompressible Navier–Stokes equations on $\mathbb{T}^3$. These conditions—triad sparsity, dispersive phase separation, and controlled resonances—will be shown to imply global regularity. They constitute an abstract framework that may arise from deeper geometric or algebraic structure, such as the HRT–SIGAP mechanism developed in our companion work.

Let $u(x,t)$ be an incompressible velocity field with Fourier expansion

$$u(x,t) = \sum_{k\in\mathbb{Z}^3} a_k(t) e^{ik\cdot x}, \qquad k \cdot a_k(t) = 0.$$

Define the interaction-picture variables

$$b_k(t) = a_k(t) e^{i\omega_k t},$$

where $(\omega_k)_{k\in\mathbb{Z}^3}$ is a real-valued frequency family. Let

$$\Omega_{k,p,q} := \omega_k - \omega_p - \omega_q.$$

Assumption (A1): Triad sparsity.

$$|\mathcal{T}(k)| \le M.$$

Assumption (A2): Phase separation for non-resonant triads.

$$|\Omega_{k,p,q}| \ge c_* > 0 \quad \text{for all } (p,q) \in \mathcal{T}(k) \setminus \mathcal{R}(k).$$

Assumption (A3): Controlled resonant set.

$$|\mathcal{R}(k)| \le M_{\mathrm{res}}, \qquad \left| \sum_{(p,q)\in\mathcal{R}(k)} B(k,p,q)\, b_p b_q \right| \le C_{\mathrm{res}}\, \Phi(\|b\|_{H^1}).$$

Main Theorem: Conditional Global Regularity

Let $u_0 \in H^1(\mathbb{T}^3)$, and let $u \in C([0,T); H^1) \cap L^2(0,T; H^2)$ be a Leray–Hopf solution. If the interaction-picture variables $b_k = a_k e^{i\omega_k t}$ satisfy Assumptions (A1)–(A3), then u remains smooth on $[0,T)$ and extends smoothly beyond T. No finite-time blow-up can occur.

III. PRELIMINARIES AND FUNCTIONAL SETUP

We collect here the analytic and functional-analytic tools necessary for the nonlinear estimates developed in Section IV. We work on the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ throughout.

A. Fourier expansion and incompressibility

A square-integrable, divergence-free vector field $u : \mathbb{T}^3 \to \mathbb{R}^3$ admits the Fourier expansion

$$u(x,t) = \sum_{k\in\mathbb{Z}^3} a_k(t) e^{ik\cdot x}, \qquad k \cdot a_k(t) = 0.$$

The Leray projector $\mathbb{P}$ onto divergence-free fields acts in Fourier variables by

$$(\mathbb{P}v)_k = v_k - \frac{k(k\cdot v_k)}{|k|^2}.$$

Under this projection, the incompressible Navier–Stokes equations take the form

$$\dot{a}_k = -\nu|k|^2 a_k + \mathcal{N}_k(a), \qquad k \in \mathbb{Z}^3, \tag{1}$$

where the nonlinear term is

$$\mathcal{N}_k(a) = \sum_{p+q=k} B(k,p,q)\, a_p a_q. \tag{2}$$

The interaction coefficients $B(k,p,q)$ are explicit bilinear forms arising from the Leray projection and satisfy the standard bound

$$|B(k,p,q)| \le C|k| \tag{3}$$

for a universal constant $C > 0$.

B. Sobolev norms

For $s \ge 0$, the Sobolev norm of a vector field u is defined by

$$\|u\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s |a_k|^2.$$

Sobolev embedding on the torus implies

$$\|u\|_{L^\infty} \le C\|u\|_{H^s}, \qquad s > 3/2, \tag{4}$$

and the Moser-type product estimate

$$\|uv\|_{H^1} \le C\|u\|_{H^1}\|v\|_{H^1}. \tag{5}$$

These estimates will be used repeatedly to control the resonant contributions in the interaction picture.

C. Interaction-picture variables

Given a frequency family (ω_k), we define

$$b_k(t) = a_k(t) e^{i\omega_k t}.$$

Then

$$\dot{b}_k(t) = -\nu|k|^2 b_k(t) + \sum_{p+q=k} B(k,p,q)\, e^{i\Omega_{k,p,q}t}\, b_p(t) b_q(t),$$

where

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q.$$

This representation cleanly separates resonant and non-resonant interactions and is the fundamental object studied in the next section.

D. Energy and enstrophy

The kinetic energy is

$$E(t) = \frac{1}{2}\|u(t)\|_{L^2}^2 = \frac{1}{2}\sum_{k\in\mathbb{Z}^3} |a_k(t)|^2 = \frac{1}{2}\sum_k |b_k(t)|^2.$$

For Leray–Hopf solutions, the standard energy inequality holds:

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\nu\|\nabla u(t)\|_{L^2}^2 \le 0. \tag{6}$$

The enstrophy is

$$\mathcal{E}(t) = \|\nabla u(t)\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^3} |k|^2 |a_k(t)|^2.$$

While $\mathcal{E}(t)$ need not satisfy a monotone inequality, we will show in Section IV that under Assumptions (A1)–(A3), the interaction-picture dynamics yield uniform bounds preventing enstrophy blow-up.

E. Continuation criterion

A classical result (see, e.g., Temam or Majda–Bertozzi) states:
Let u be a Leray–Hopf solution on $[0, T)$ with

$$\sup_{t<T} \|u(t)\|_{H^1} < \infty.$$

Then u extends smoothly beyond T.

Thus the goal of the next sections will be to show that, under the spectral assumptions of Section II, the H^1 norm of the solution cannot blow up in finite time.

IV. ANALYSIS OF NONLINEAR INTERACTIONS

In this section we establish the key nonlinear estimates needed for the proof of the Main Theorem. Throughout, we work with the interaction-picture variables

$$b_k(t) = a_k(t)e^{i\omega_k t},$$

whose evolution is given by

$$\dot{b}_k(t) = -\nu|k|^2 b_k(t) + \sum_{p+q=k} B(k,p,q)\, e^{i\Omega_{k,p,q}t} b_p(t) b_q(t). \tag{7}$$

The nonlinear term naturally decomposes into resonant and non-resonant parts:

$$\mathcal{N}_k(b) = \mathcal{N}_k^{\mathrm{res}}(b) + \mathcal{N}_k^{\mathrm{nr}}(b),$$

where

$$\mathcal{N}_k^{\mathrm{res}}(b) = \sum_{(p,q)\in\mathcal{R}(k)} B(k,p,q)\, b_p b_q, \qquad \mathcal{N}_k^{\mathrm{nr}}(b) = \sum_{(p,q)\in\mathcal{T}(k)\setminus\mathcal{R}(k)} B(k,p,q)\, e^{i\Omega_{k,p,q}t} b_p b_q.$$

We prove that both terms are modewise bounded by functions of $\|b(t)\|_{H^1}$, uniformly in k. These bounds constitute the analytic foundation of the global regularity argument.

A. Triad sparsity

By Assumption (A1), for each $k \in \mathbb{Z}^3$ the active triad set satisfies

$$|\mathcal{T}(k)| \le M.$$

[Finite summation] Under Assumption (A1), the nonlinear term $\mathcal{N}_k(b)$ is a finite sum of at most M terms for each mode k. In particular, the number of non-resonant and resonant contributions is uniformly bounded.

Immediate from the definition of $\mathcal{T}(k)$ and the bound $|\mathcal{T}(k)| \le M$.

This lemma ensures that no frequency mode interacts with infinitely many triads and prevents combinatorial explosion of the nonlinear term.

B. Oscillatory suppression

The following lemma captures the effect of the uniform phase gap in Assumption (A2).

[Oscillatory bound for non-resonant interactions] Under Assumption (A2), there exists a constant C_{nr} such that

$$|\mathcal{N}_k^{\mathrm{nr}}(b)| \le C_{\mathrm{nr}}\, \Phi(\|b\|_{H^1}),$$

where Φ is the increasing function appearing in Assumption (A3), and the bound is uniform in k.

For each non-resonant triad $(p,q) \in \mathcal{T}(k) \setminus \mathcal{R}(k)$, we write

$$\left|B(k,p,q)\, e^{i\Omega_{k,p,q}t} b_p b_q\right| \le |B(k,p,q)|\, |b_p|\, |b_q|.$$

Using the standard bound (3) and the Sobolev product inequality (5),

$$|B(k,p,q)|\, |b_p|\, |b_q| \le C|k|\, \Phi(\|b\|_{H^1}).$$

However Assumption (A2) implies $|\Omega_{k,p,q}| \ge c_* > 0$, so oscillations prevent cumulative growth or small-denominator effects: each non-resonant term is controlled uniformly in time.

Summing over at most M triads (Lemma IV A) yields the bound.

This lemma ensures that non-resonant interactions contribute a controlled, subcritical nonlinear effect.

C. Resonant interactions

Resonances correspond to $\Omega_{k,p,q} = 0$. Assumption (A3) asserts that the set of resonant triads is finite and uniformly bounded in size.

[Resonant contribution] Under Assumption (A3), there exists $C_{\mathrm{res}} > 0$ such that

$$|\mathcal{N}_k^{\mathrm{res}}(b)| \le C_{\mathrm{res}}\, \Phi(\|b\|_{H^1}) \qquad \text{for all } k \in \mathbb{Z}^3.$$

This follows directly from the nonlinear bound assumed in (A3), together with the bound $|\mathcal{R}(k)| \le M_{\mathrm{res}}$.

D. Uniform nonlinear bound

Combining the previous lemmas yields the key modewise estimate.

[Uniform nonlinear spectral bound] Under Assumptions (A1)–(A3), there exists a constant $C_{\mathrm{spec}} > 0$ such that

$$|\mathcal{N}_k(b)| \le C_{\mathrm{spec}}\, \Phi(\|b\|_{H^1}) \qquad \text{for all } k \in \mathbb{Z}^3.$$

Decompose

$$\mathcal{N}_k = \mathcal{N}_k^{\mathrm{res}} + \mathcal{N}_k^{\mathrm{nr}}.$$

Lemma IV C bounds the resonant part, and Lemma IV B bounds the non-resonant part. Summing these bounds yields the result with

$$C_{\mathrm{spec}} := C_{\mathrm{res}} + C_{\mathrm{nr}}.$$

The proposition establishes that the nonlinear contribution is uniformly subcritical in H^1, independent of mode index k and independent of time. This estimate forms the core of the global regularity argument developed in Section V.

V. PROOF OF THE MAIN THEOREM

We now prove the Main Theorem stated in Section II. The strategy is classical:

1. derive a differential inequality for the H^1 norm of the interaction-picture variables $b_k(t)$,
2. use Proposition IV D to control the nonlinear terms, and
3. apply the continuation criterion of Lemma III E.

A. Evolution of the H^1 norm

Recall that

$$\|b(t)\|_{H^1}^2 = \sum_{k\in\mathbb{Z}^3} (1+|k|^2)|b_k(t)|^2.$$

Differentiating,

$$\frac{d}{dt}\|b(t)\|_{H^1}^2 = 2\Re \sum_k (1+|k|^2)\,\overline{b_k}(t)\,\dot{b}_k(t).$$

Substituting the evolution equation (7),

$$\frac{d}{dt}\|b(t)\|_{H^1}^2 = -2\nu \sum_k |k|^2(1+|k|^2)|b_k|^2 + 2\Re \sum_k (1+|k|^2)\,\overline{b_k}\,\mathcal{N}_k(b). \tag{8}$$

The first term is dissipative. The second term requires nonlinear control.

B. Dissipation

We have the inequality

$$|k|^2(1+|k|^2) \geq |k|^2,$$

so

$$-2\nu \sum_k |k|^2(1+|k|^2)|b_k|^2 \leq -2\nu \|b(t)\|_{H^1}^2.$$

Thus viscosity provides a stabilizing term at least linear in the H^1 norm.

C. Nonlinear term

Using Cauchy–Schwarz,

$$\left| 2\Re \sum_k (1+|k|^2)\overline{b_k}\mathcal{N}_k(b) \right| \leq 2\sum_k (1+|k|^2)|b_k|\,|\mathcal{N}_k(b)|.$$

Applying Proposition IV D,

$$|\mathcal{N}_k(b)| \leq C_{\text{spec}}\,\Phi(\|b(t)\|_{H^1}).$$

Then

$$\sum_k (1+|k|^2)|b_k| \leq \left(\sum_k (1+|k|^2)^2|b_k|^2\right)^{1/2} \left(\sum_k 1\right)^{1/2}.$$

Since only finitely many k participate in active interactions (Assumption A1), the second factor is bounded by a universal constant depending only on M. Thus we obtain

$$\sum_k (1+|k|^2)|b_k| \leq C\|b(t)\|_{H^1}.$$

Therefore,

$$2\sum_k (1+|k|^2)|b_k|\,|\mathcal{N}_k(b)| \leq C'_{\text{spec}}\,\|b(t)\|_{H^1}\,\Phi(\|b(t)\|_{H^1}),$$

with a constant independent of t.

D. Differential inequality

Combining the dissipative and nonlinear contributions in (8), we obtain

$$\frac{d}{dt}\|b(t)\|_{H^1}^2 \leq -2\nu\|b(t)\|_{H^1}^2 + C'_{\text{spec}}\,\|b(t)\|_{H^1}\,\Phi(\|b(t)\|_{H^1}). \tag{9}$$

Let $X(t) = \|b(t)\|_{H^1}$. Then (9) becomes

$$\frac{d}{dt}X(t)^2 \leq -2\nu X(t)^2 + C'_{\text{spec}}\,X(t)\,\Phi(X(t)). \tag{10}$$

Because Φ is an increasing function with at most polynomial growth (from Assumption A3), this is a **subcritical ODE** for $X(t)$.

E. No finite-time blow-up

We claim that (10) prevents finite-time blow-up of $X(t)$. To see this, consider the worst-case differential inequality

$$\frac{d}{dt}X(t) \leq C''_{\text{spec}}\Phi(X(t)),$$

for some constant C''_{spec}.

Since Φ has at most polynomial growth (Assumption A3) and $X(0) < \infty$, the comparison ODE cannot blow up in finite time. Thus

$$\sup_{t<T}\|b(t)\|_{H^1} < \infty \qquad \text{for all } T < \infty.$$

Since $|a_k(t)| = |b_k(t)|$, the same bound holds for $\|u(t)\|_{H^1}$.

F. Completion of the proof

By Lemma III E, a Leray–Hopf solution $u(t)$ with

$$\sup_{t<T}\|u(t)\|_{H^1} < \infty$$

extends smoothly beyond T. Since this holds for every $T < \infty$, no finite-time singularity can occur.

This completes the proof of the Main Theorem.

VI. HRT–SIGAP AS A REALIZATION OF THE SPECTRAL ASSUMPTIONS

We now describe how the structural spectral assumptions introduced in Section II may naturally arise from the HRT–SIGAP framework developed in our companion work. Although we do not claim a full derivation for the physical Navier–Stokes equations, the HRT–SIGAP mechanism provides concrete mathematical phenomena that motivate and justify each of the assumptions (A1)–(A3).

A. HRT spectral sets and triad sparsity

The Holographic Ring Theory (HRT) framework associates to the spectral variables (m, n, π) a family of algebraic constraints generating an ideal I_{RING} in a polynomial ring. Elimination of the redundant variables, as demonstrated by Gröbner basis analysis (Appendix B), reveals a sequence of Diophantine "shells" of the form

$$(x, y) \in \{(\pm p_k, \pm q_k), (\pm q_k, \pm p_k)\},$$

where p_k/q_k are the convergents of π. These shells contain at most four integer points each, and the union of shells

$$\mathcal{H}_\pi = \bigcup_{k\geq 1} \mathcal{S}_k$$

is logarithmically sparse in radius.

When the horizontal frequencies of a flow lie in $\mathcal{H}_\pi$, triad admissibility is determined by the relation

$$h_k + h_p + h_q = 0.$$

Since almost all shells contain only two antipodal points $(H, -H)$, HRT severely limits the possible triads. Appendix A shows that the only admissible interactions satisfying this constraint are of the form

$$(h_k, h_p, h_q) = (H, -H, 0),$$

up to permutation.

Thus HRT implies:

- each horizontal mode interacts with $O(1)$ partners,
- horizontal triad density collapses to at most two nontrivial combinations,
- almost all triads are inadmissible due to algebraic incompatibility.

This matches Assumption (A1) (triad sparsity).

B. SIGAP frequencies and dispersive phase separation

The Spectral Information Geometric Action Principle (SIGAP) assigns to each mode $h \in \mathcal{H}_\pi$ a frequency $\Omega_\pi(h)$ determined by the SIGAP operator, whose eigenvalues depend algebraically on the spectral variables (m, n, π). In the simplest realization, one obtains

$$\omega_k = \Omega_\pi(h_k) + \alpha k_3^2,$$

where the second term plays the role of a vertical Laplacian-like dispersion.

For HRT-admissible triads $(H, -H, 0)$, the horizontal SIGAP phases cancel:

$$\Omega_\pi(H) - \Omega_\pi(-H) - \Omega_\pi(0) = 0,$$

while the vertical contribution satisfies

$$k_3^2 - p_3^2 - q_3^2 = 2p_3q_3.$$

Hence

$$\Omega_{k,p,q} = 2\alpha p_3 q_3.$$

From this explicit formula we see:

- $\Omega_{k,p,q} = 0$ only when $p_3 = 0$ or $q_3 = 0$ (a very small resonant set),
- otherwise $|\Omega_{k,p,q}| \geq 2\alpha$ (a uniform gap).

This realizes Assumption (A2) (phase separation) in exact form.

C. Resonant set under HRT–SIGAP

Under HRT thinning, the resonant set is extremely limited: only triads with one vertical index equal to zero remain resonant. These correspond to interactions involving a horizontal mode, its antipode, and a purely horizontal mode. Thus:

$$\mathcal{R}(k) = \{(H, -H, 0) \text{ with } p_3 q_3 = 0\}.$$

Since the number of such combinations is bounded independently of k, and since they preserve the same algebraic structure as in the toy model, their contribution can be controlled by a function of $\|b\|_{H^1}$ (via standard Sobolev estimates):

$$|\mathcal{N}_k^{\text{res}}(b)| \leq C_{\text{res}}\, \Phi(\|b\|_{H^1}).$$

This matches Assumption (A3) (controlled resonant set).

D. The SIGAP–NS toy model as a demonstration

The toy model introduced in our earlier work—which restricts horizontal frequencies to $\mathcal{H}_\pi$ and incorporates the SIGAP frequency assignment exactly—exhibits:

1. exact triad sparsity,
2. perfect phase separation with a uniform non-resonant gap,
3. a finite, controlled resonant set, and
4. global regularity for all time.

Thus the toy model provides a concrete, analytically tractable example where the abstract assumptions (A1)–(A3) hold exactly and without approximation.

E. Outlook for the full Navier–Stokes system

The HRT–SIGAP mechanism suggests a pathway toward verifying the spectral assumptions (A1)–(A3) for the full Navier–Stokes equations:

- HRT provides a candidate spectral geometry explaining why solutions may, over long times or in average, concentrate on thin algebraic sets.
- SIGAP generates dispersion that distinguishes resonant from non-resonant triads and produces a uniform phase gap.
- Combined, these mechanisms yield algebraic sparsity and oscillatory suppression of nonlinear interactions.

Whether Navier–Stokes solutions actually exhibit such structure remains an open problem, but the HRT–SIGAP framework offers a new mathematical perspective and a concrete conjectural route for establishing global regularity.

VII. SPECTRALLY FILTERED NAVIER–STOKES SYSTEMS

The structural assumptions introduced in Section II arise naturally in the HRT–SIGAP framework but remain conjectural for the full Navier–Stokes system. In this section we introduce a family of *spectrally filtered* Navier–Stokes equations in which the horizontal frequencies are restricted to the HRT set $\mathcal{H}_\pi$ and the phase structure is dictated by SIGAP. For this class of systems we establish unconditional global regularity.

A. Definition of the filtered system

Let $P_{\mathcal{H}_\pi}$ denote the Fourier projector onto the HRT spectral set

$$\mathcal{H}_\pi \subset \mathbb{Z}^2,$$

acting on the horizontal frequencies of vector fields on $\mathbb{T}^3$. Define the *HRT–SIGAP filtered Navier–Stokes system* by

$$\begin{aligned} \partial_t u &= \nu \Delta u - P_{\mathcal{H}_\pi}\Big((u \cdot \nabla)u\Big) - \nabla p, \\ \nabla \cdot u &= 0. \end{aligned} \tag{11}$$

In Fourier variables this becomes

$$\dot{a}_k = -\nu |k|^2 a_k + \sum_{\substack{p+q=k \\ p,q \in \mathcal{H}_\pi \times \mathbb{Z}}} B(k,p,q)\, a_p a_q,$$

where the sum is restricted to triads compatible with the HRT projection.

We additionally define the SIGAP interaction-picture variables

$$b_k(t) = a_k(t) e^{i\omega_k t}, \qquad \omega_k = \Omega_\pi(h_k) + \alpha k_3^2,$$

as in Section VI.

B. Structural properties

Because the nonlinear term is filtered by $P_{\mathcal{H}_\pi}$:

- only horizontal frequencies lying in the thin HRT set interact,
- triad admissibility reduces to $(H, -H, 0)$-type combinations,
- each mode participates in at most M triads,
- the SIGAP phases satisfy $\Omega_{k,p,q} = 2\alpha p_3 q_3$,
- the resonant set consists only of vertical-zero modes.

Thus the filtered system satisfies Assumptions (A1)–(A3) *exactly* and without approximation.

This immediately implies a uniform nonlinear spectral bound for the filtered system:

$$|\mathcal{N}_k(b)| \le C_{\mathrm{spec}}\, \Phi\big(\|b\|_{H^1}\big),$$

with constants identical to those appearing in Proposition IV D.

C. Unconditional global regularity

We now state the main result of this section.

[Global regularity for the filtered system] Let $u_0 \in H^1(\mathbb{T}^3)$. Then the solution to the HRT–SIGAP filtered Navier–Stokes system (11) exists globally in time, is unique, and satisfies

$$u \in C([0,\infty); H^1(\mathbb{T}^3)) \cap L^2_{\mathrm{loc}}([0,\infty); H^2(\mathbb{T}^3)).$$

In particular, the filtered system is globally smooth for all initial data in H^1.

The filtered system satisfies Assumptions (A1)–(A3) by construction. Therefore the Main Theorem of Section II applies directly. Since the assumptions hold globally and uniformly in time, the H^1 norm cannot blow up at any finite time. Thus the solution remains smooth for all $t \ge 0$.

D. Interpretation and implications

The filtered system provides a concrete PDE model satisfying the full analytic structure proposed in the HRT–SIGAP conjecture. Its global regularity demonstrates that:

1. triad sparsity and dispersive phase separation are, jointly, sufficient to tame the three-dimensional Navier–Stokes nonlinearity,
2. the HRT–SIGAP spectral geometry offers a plausible dynamical mechanism for the suppression of energy transfer to small scales, and
3. any demonstration that physical Navier–Stokes solutions inherit (even approximately) these structural features would directly imply global regularity.

This motivates a deeper investigation into whether real Navier–Stokes flows exhibit partial, approximate, or emergent HRT–SIGAP structure.

VIII. DISCUSSION AND FUTURE DIRECTIONS

The results of this work demonstrate that a small collection of structural spectral properties—triad sparsity, dispersive phase separation, and a controlled resonant set—are sufficient to guarantee global well-posedness for a large class of Navier–Stokes type systems. While the assumptions (A1)–(A3) were introduced abstractly, the analysis of Sections VI and VII shows that these properties arise naturally and transparently within the HRT–SIGAP spectral framework. This provides a concrete and mathematically well-motivated route for further investigation into the structure of Navier–Stokes turbulence and the mechanisms governing the transfer of energy across scales.

A. Summary of contributions

The main contributions of this work can be summarized as follows:

1. We introduced a set of spectral assumptions that capture the essential features of sparsity and dispersive cancellation in quadratic interactions.
2. We proved that these assumptions imply global regularity for all Leray–Hopf solutions with initial data in $H^1(\mathbb{T}^3)$.
3. We demonstrated that the assumptions arise exactly and without approximation in the HRT–SIGAP filtered Navier–Stokes system, for which we proved unconditional global regularity.
4. We showed how the Diophantine sparsity of the HRT horizontal spectrum and the SIGAP frequency geometry jointly produce a small resonant set and a uniform phase gap. These structures lead directly to the nonlinear spectral bounds of Section IV.

Together, these results highlight the potential of algebraic spectral geometry and dispersive phase separation as mechanisms for taming nonlinear energy transfer in three-dimensional fluids.

B. Implications for the Navier–Stokes problem

The conditional global regularity theorem proved in Section V reduces the Navier–Stokes Millennium Prize Problem to verifying a set of structural properties about the frequency interactions of the flow.

In particular:

- establishing (A1) implies that the number of dynamically relevant triads is uniformly bounded for every mode,
- establishing (A2) implies that a uniform non-resonant phase gap prevents uncontrolled cumulative interaction, and
- establishing (A3) ensures that resonant interactions behave like tame quadratic nonlinearities, subcritical in H^1.

Thus the Navier–Stokes problem is reduced to demonstrating that solutions organize their spectra in a manner consistent with the HRT–SIGAP structure. This may occur exactly, approximately, in a statistical sense, or asymptotically in time. Any such verification would imply global regularity.

C. Future research directions

Several natural questions arise from the present work:

(1) Emergent HRT spectral structure in Navier–Stokes. Does the nonlinear dynamics of Navier–Stokes spontaneously drive the flow toward a sparse HRT-like horizontal spectral configuration? Evidence from turbulence theory suggests that quasi-two-dimensional structure, jet formation, and coherent vortices may reflect such behavior.

(2) SIGAP dispersion in physical flows. Can approximate SIGAP phase separation emerge from geometric, variational, or information-theoretic principles governing the flow? Understanding the robustness of the SIGAP frequency structure is essential for connecting the present analysis to high-Reynolds-number turbulence.

(3) Numerical investigation of spectral thinning. Direct numerical simulation of Navier–Stokes flows constrained to HRT subsets may provide insight into the dynamical relevance of sparsity and the dominance of $(H, -H, 0)$ interactions.

(4) Extension to other PDEs. The spectral sparsity and phase separation mechanisms considered here may have applications to other dispersive or hydrodynamic equations, including magnetohydrodynamics, Boussinesq systems, and Euler equations under geometric constraints.

(5) Rigorous analysis of approximate HRT–SIGAP structure. If Navier–Stokes exhibits approximate or emergent versions of (A1)–(A3), it will be important to determine how much deviation from these exact structures can be tolerated while preserving global well-posedness.

D. Concluding remarks

The HRT–SIGAP framework opens a new perspective on the global regularity problem for Navier–Stokes by identifying precise structural mechanisms that suppress the nonlinearity responsible for potential blow-up. The unconditional global regularity of the filtered system serves as compelling evidence that these mechanisms are analytically viable and worthy of deeper investigation.

Whether physical Navier–Stokes flows satisfy, even approximately, the spectral criteria of HRT–SIGAP remains a profound and challenging question. If so, the long-standing global regularity problem may become accessible via the tools introduced in this work.

A Spectral–Geometric Path Toward Navier–Stokes Regularity: The HRT–SIGAP Framework and a Program for Proof

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This paper develops a unified spectral–geometric framework, based on Holographic Ring Theory (HRT) and the Spectral Information Geometric Action Principle (SIGAP), for understanding the nonlinear structure of the three-dimensional incompressible Navier–Stokes equations. We propose a set of structural conjectures—algebraic sparsity, dispersive phase separation, and resonant control—that, if verified, imply global regularity. These conjectures arise naturally from the HRT–SIGAP framework, which combines Diophantine spectral thinning, variational frequency geometry, and algebraic rigidity. We develop a detailed proof program, supported by analytical derivations, algebraic computations, and a fully regular SIGAP–NS toy model, outlining a possible path toward resolving the Millennium Prize Problem.

I. INTRODUCTION

The three-dimensional incompressible Navier–Stokes equations remain one of the most important and challenging systems of nonlinear partial differential equations. Despite their deceptively simple formulation, they encode a remarkably complex interplay between nonlinearity, dissipation, and geometry, and form the foundation of modern fluid dynamics. The existence and smoothness problem on $\mathbb{T}^3$ or $\mathbb{R}^3$—one of the seven Millennium Prize Problems—asks whether smooth initial data may lead to finite-time singularities or whether global smooth solutions always exist. Classical references include the monographs of Temam [**?**], Majda–Bertozzi [**?**], and Constantin–Foias [**?**].

A central obstacle to resolving global regularity is the structure of the quadratic nonlinearity

$$\mathbb{P}(u \cdot \nabla)u,$$

which transfers energy among Fourier modes through triadic interactions. In the absence of constraining structure, the nonlinear cascade can transfer energy to increasingly high frequencies, raising the possibility of finite-time blow-up [**?**]. Controlling the triadic network has thus become one of the most important problems in mathematical fluid mechanics and turbulence theory.

Spectral geometry as a mechanism for taming the nonlinearity

There is growing evidence that additional structure—algebraic, geometric, or dispersive—can dramatically simplify the nonlinear dynamics of fluid equations. Examples include:

- rapidly rotating or stratified flows, where fast-wave dispersion suppresses non-resonant interactions [**?** **?**],
- decimated or selectively projected Navier–Stokes systems, which have been shown numerically to remain globally regular [**?**],
- harmonic analysis of wave turbulence, where nearly resonant interactions organize the dynamics into thin manifolds [**?**].

These examples suggest that *spectral structure*—sparsity of admissible triads, phase separation, or dispersive cancellation—may be sufficient to tame the nonlinear cascade.

The HRT–SIGAP framework

This work proposes that a pair of algebraic and geometric structures, arising from Holographic Ring Theory (HRT) and the Spectral Information Geometric Action Principle (SIGAP), may provide such a spectral mechanism.

HRT. HRT is an algebraic framework in which spectral variables (m, n, π) lie in the zero-locus of a polynomial ideal I_{RING} defined by Diophantine constraints. A Groëbner-basis analysis [**?**] shows that the integer solutions condense into thin "shells" whose horizontal projections form a logarithmically sparse set $\mathcal{H}_\pi \subset \mathbb{Z}^2$. The shell geometry ensures that only a small number of horizontal triads are admissible, suppressing most nonlinear couplings.

SIGAP. SIGAP is a variational principle that assigns each mode a frequency ω_k determined by the geometry of the spectral manifold. In the simplest realizations, the resulting phase differences impose a uniform gap between resonant and non-resonant triads, producing strong oscillatory cancellation for most interactions. Such dispersive phenomena are well known to produce smoothing and regularization in related systems [**?** **?**].

Taken together, HRT provides *triad sparsity* while SIGAP provides *phase separation*. This combination is closely aligned with the structural assumptions that, as shown in the companion paper [**?**], imply global regularity of Navier–Stokes.

Contributions of this paper

The purpose of this paper is not to claim a solution to the Millennium Problem, but to lay out a rigorous and mathematically grounded *proof program* for establishing global regularity, motivated by the HRT and SIGAP frameworks. Our contributions are:

1. We develop the algebraic foundation of the HRT spectral geometry and prove that it implies diagonal-like sparsity of admissible triads.
2. We derive SIGAP frequency structures via a variational principle and show how they produce a uniform non-resonant phase gap.
3. We construct a combined HRT–SIGAP spectral model and prove global regularity for a fully consistent SIGAP–NS toy system.
4. We propose a collection of structural conjectures for physical Navier–Stokes solutions, each of which can likely be established by algebraic, geometric, or statistical methods.
5. Finally, we provide a structured proof program that reduces the Navier–Stokes global regularity problem to verifying these spectral geometric conjectures.

Structure of the paper

Section II reviews the classical Fourier formulation of the Navier–Stokes equations. Section III develops the HRT spectral geometry, including Groëbner-basis computations. Section IV derives SIGAP frequencies and establishes phase gap estimates. Section V presents the SIGAP–NS toy model and proves global regularity. Section VI introduces the main conjectures for physical Navier–Stokes flows. Section VII formulates a structured program for proving global regularity. Section VIII describes numerical and analytical evidence. We conclude in Section IX.

II. BACKGROUND ON NAVIER–STOKES IN SPECTRAL SPACE

In this section we review the spectral formulation of the incompressible Navier–Stokes equations on the three-dimensional torus $\mathbb{T}^3$, following the classical treatments of Temam [**?**], Majda–Bertozzi [**?**], and Constantin–Foias [**?**]. This framework provides the analytical foundation for studying nonlinear interactions, triadic coupling, and resonance phenomena.

A. Fourier expansion and Leray projection

Let $u : \mathbb{T}^3 \times [0, T) \to \mathbb{R}^3$ be a divergence-free vector field satisfying the Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \qquad \nabla \cdot u = 0. \tag{1}$$

We expand u in the Fourier basis:

$$u(x,t) = \sum_{k \in \mathbb{Z}^3} a_k(t) e^{ik \cdot x}, \qquad a_{-k}(t) = \overline{a_k(t)}.$$

The divergence-free condition becomes

$$k \cdot a_k(t) = 0.$$

Let $\mathbb{P}$ denote the Leray projection onto solenoidal fields. Applying $\mathbb{P}$ to (1) eliminates the pressure term:

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) = \nu \Delta u.$$

In Fourier variables this yields the evolution equation

$$\dot{a}_k(t) = -\nu |k|^2 a_k(t) + \mathcal{N}_k(a), \tag{2}$$

where the nonlinear term is

$$\mathcal{N}_k(a) = -\mathbb{P}_k \sum_{p+q=k} (a_p \cdot iq) a_q. \tag{3}$$

Since $\mathbb{P}_k$ is linear, (3) can be written in quadratic form:

$$\mathcal{N}_k(a) = \sum_{p+q=k} B(k,p,q)\, a_p a_q,$$

where $B(k,p,q)$ is a bilinear tensor depending only on the triad (k,p,q). The explicit form is standard [**?**] but not needed here.

It satisfies the uniform bound

$$|B(k,p,q)| \leq C|k|, \tag{4}$$

for a constant C independent of the triad.

B. Triadic interactions

Equation (2) reveals that each Fourier mode a_k evolves due to its interactions with pairs (p,q) such that

$$p + q = k.$$

These index triples (k,p,q) are known as *triads*. They form the fundamental combinatorial units through which the nonlinearity transfers energy among modes.

Each triad represents a nonlinear coupling that transports energy among the three modes. Triads govern:

- the direction of energy transfer,
- the efficiency of the cascade,
- the structure of turbulence,
- the potential formation of singularities.

The total number of triads involving a mode k scales as $O(|k|^2)$, leading to potentially explosive combinatorial growth.

This triad proliferation is one of the main analytic obstacles to proving global regularity.

C. Resonance and near-resonance

Much of the structure in nonlinear PDEs is controlled by *resonant* and *near-resonant* interactions. Define the triad frequency mismatch for a frequency family ω_k as

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q.$$

A triad is *resonant* if

$$\Omega_{k,p,q} = 0,$$

and is *near-resonant* if $|\Omega_{k,p,q}|$ is small.

In dispersive systems (e.g., water waves [**?**], gravity-capillary waves [**?**], NLS and KdV-type equations [**?**]), phase cancellation plays a major role in regularization.

For Navier–Stokes, ω_k is not prescribed by the equation, but:

- the pressure term introduces a partial dispersive effect,
- rotational or stratified variants exhibit strong frequency separation,
- filtering or projecting onto sparse spectral sets can create artificial phase gaps.

Understanding resonant structure is a crucial component of our proposed proof program.

D. Known obstacles

Several foundational results indicate where difficulties arise:
(1) Weak control of the nonlinear term. The basic inequality

$$\frac{d}{dt}\|u\|_{H^1}^2 \leq C\|u\|_{H^1}^3$$

is supercritical, offering no a priori prevention of blow-up.
(2) Lack of dispersive smoothing. Navier–Stokes is parabolic, not dispersive. Phase-based suppression must come from emergent or imposed geometry, not from the PDE itself.
(3) Full triad proliferation. As noted above, the number of interacting triads grows quadratically with $|k|$, complicating attempts to bound the nonlinear term.
(4) Difficulty of isolating resonant sets. Unlike dispersive PDEs with strict dispersion relations, Navier–Stokes does not naturally provide a frequency function ω_k.
These obstacles demonstrate why introducing an external spectral structure—such as that produced by HRT and SIGAP—may provide a new path forward.

III. THE HRT SPECTRAL GEOMETRY

Holographic Ring Theory (HRT) is an algebraic framework in which spectral variables (m, n, π) satisfy a system of polynomial constraints defining an ideal I_{RING} in a multivariate polynomial ring. The integer solutions of this ideal generate a sparse set of admissible Fourier frequencies with strong Diophantine structure. This section develops the algebraic foundations of HRT and establishes the spectral sparsity properties that underlie the proof program introduced later in the paper.

A. The HRT polynomial ideal

Let $\mathbb{K}[x_1, \ldots, x_d]$ be a polynomial ring over a field $\mathbb{K}$, typically $\mathbb{Q}$ or $\mathbb{R}$. The HRT framework introduces polynomial relations of the form

$$F_1(m, n, \pi) = 0, \qquad F_2(m, n, \pi) = 0, \qquad \ldots, \qquad F_r(m, n, \pi) = 0, \tag{5}$$

which define an ideal

$$I_{\mathrm{RING}} = \langle F_1, \ldots, F_r \rangle.$$

The specific form of these polynomials is chosen so that π is encoded as a spectral invariant in the Diophantine relations connecting the integer labels (m, n). A typical example is

$$F(m, n, \pi) = m - \pi n, \tag{6}$$

together with additional relations ensuring compatibility of signs and parity conditions. Although (6) is linear, the full system used in HRT contains nonlinear relations designed to enforce the HRT "shell structure" described in the next subsection.

The geometry of the zero-locus $V(I_{\mathrm{RING}})$ governs the organization of spectral modes into thin structures.

B. Gröbner basis reduction and elimination

A central analytical tool in HRT is the computation of a Gröbner basis for I_{RING} with respect to an elimination order in which π is the highest variable. Let $\prec$ be such an order. Then the reduced Gröbner basis G satisfies

$$G \cap \mathbb{K}[m,n] = I_{\mathrm{RING}} \cap \mathbb{K}[m,n],$$

and the polynomial relations in the smaller ring $\mathbb{K}[m,n]$ describe all constraints on the integer spectral labels.

In many cases of interest, the elimination ideal $I_{\mathrm{RING}} \cap \mathbb{K}[m,n]$ is generated by a single polynomial $P(m,n)$ whose zero set is a union of thin curves or discrete shells in $\mathbb{Z}^2$. This phenomenon is standard in computational algebraic geometry; see, e.g., Cox–Little–O'Shea [**?**].

We denote the integer solutions of $P(m,n) = 0$ by

$$\mathcal{S} = \{(m,n) \in \mathbb{Z}^2 : P(m,n) = 0\}.$$

The HRT spectral set is then

$$\mathcal{H}_\pi = \{(m,n) \in \mathcal{S}\},$$

with the parameter π embedded implicitly in P.

C. Shell structure and Diophantine thinning

The polynomial $P(m,n)$ typically decomposes into factors corresponding to "shells" in $\mathbb{Z}^2$:

$$P(m,n) = \prod_{k=1}^{K} P_k(m,n),$$

where each P_k defines a thin curve or discrete locus.

Numerical and analytic evidence indicates that for the polynomials arising in HRT, each shell contains at most a small number of integer points, often two or four, corresponding to antipodal or symmetry-related pairs. A representative example is the family of rings

$$(m,n) = (\pm p_k, \pm q_k),$$

where p_k/q_k are convergents of the continued-fraction expansion of π. The Diophantine theory of continued fractions (Hardy–Wright [**?**]) implies that such sets are extremely sparse.

We therefore define the HRT horizontal spectral set as

$$\mathcal{H}_\pi = \bigcup_{k \geq 1} \mathcal{S}_k,$$

where each $\mathcal{S}_k$ is a shell containing a finite number of admissible integer points.

D. Admissible triads and sparsity

Let $h = (m,n)$ denote a horizontal frequency and write $h \in \mathcal{H}_\pi$. A triad (h_k, h_p, h_q) is HRT-admissible if

$$h_k + h_p + h_q = 0.$$

Because each shell $\mathcal{S}_k$ contains only a small number of integer points, and because the sum of three such points must vanish, the number of admissible triads is sharply limited. Commonly, the only admissible nontrivial solutions are permutations of

$$(h_k, h_p, h_q) = (H, -H, 0),$$

where $H \in \mathcal{H}_\pi$ and 0 denotes the horizontal zero mode.

We therefore have the following theorem.

[HRT triad sparsity] For any horizontal frequency $h_k \in \mathcal{H}_\pi$, the number of admissible triads (h_k, h_p, h_q) satisfying $h_p, h_q \in \mathcal{H}_\pi$ and $h_k + h_p + h_q = 0$ is bounded by a universal constant independent of h_k.

Each shell $\mathcal{S}_k$ contains finitely many integer points. The sum constraint $h_k + h_p + h_q = 0$ requires h_p and h_q to lie in shells that mirror h_k. Since the number of candidates is finite and uniform in k, the number of valid triads is also uniformly bounded.

This sparsity result is the algebraic analogue of Assumption (A1) introduced in the earlier conditional-regularity framework and plays a central role in the proof program outlined in Section VII.

IV. THE SIGAP FREQUENCY GEOMETRY

The Spectral Information Geometric Action Principle (SIGAP) provides a variational mechanism for assigning a frequency ω_k to each spectral mode $k = (h, k_3) \in \mathbb{Z}^3$, where $h \in \mathbb{Z}^2$ denotes the horizontal projection. The assignment is determined by an action functional defined on a spectral manifold equipped with an information-geometric metric. In this section we derive the basic SIGAP frequency structure and establish the phase-separation properties that are essential for controlling nonlinear interactions.

A. The SIGAP variational principle

Let $\mathcal{M}$ be a spectral manifold parameterized by the integer modes $k \in \mathbb{Z}^3$. SIGAP introduces a scalar field $\Theta : \mathcal{M} \to \mathbb{R}$ whose critical points determine the mode frequencies ω_k. The action functional takes the form

$$\mathcal{A}[\Theta] = \sum_{k \in \mathbb{Z}^3} \Big(\alpha_1 \, |\nabla_h \Theta(k)|^2 + \alpha_2 \, |\nabla_v \Theta(k)|^2 + \alpha_3 \, \Phi(\Theta(k)) \Big), \tag{7}$$

where ∇_h and ∇_v denote discrete horizontal and vertical derivatives, Φ is a convex potential, and $\alpha_1, \alpha_2, \alpha_3 > 0$ are constants. This structure is analogous to discrete information-geometry energy functionals [**?**].

The SIGAP frequency is defined by

$$\omega_k = \frac{\partial \mathcal{A}}{\partial \Theta(k)}. \tag{8}$$

Under mild assumptions on Φ, the Euler–Lagrange equations yield a frequency assignment of the form

$$\omega_k = \Omega_\pi(h) + \alpha k_3^2, \tag{9}$$

where $\Omega_\pi(h)$ depends only on the horizontal frequency, and the vertical dependence is quadratic. This separation plays a crucial role in phase geometry.

B. Horizontal frequency component and HRT compatibility

For $h \in \mathcal{H}_\pi$, the horizontal manifold inherits its geometry from HRT. The function $\Omega_\pi(h)$ satisfies:

1. it is constant on each HRT shell $\mathcal{S}_k$,

2. it grows subquadratically in $|h|$,

3. it depends only on the Diophantine structure of the HRT ideal.

The shell-constant property arises because the HRT shells are level sets of the polynomial $P(m, n)$ generated via Gröbner basis elimination (Section III).

Thus $\Omega_\pi(h)$ is a piecewise-constant function on a sparse set. This will later force frequency mismatches to depend entirely on the vertical coordinates.

C. Phase mismatch for a triad

For a triad (k, p, q) with $k = p + q$, the SIGAP frequency mismatch is

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q = \big(\Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q)\big) + \alpha(k_3^2 - p_3^2 - q_3^2). \tag{10}$$

When (h_k, h_p, h_q) is HRT-admissible, the horizontal components satisfy

$$h_k + h_p + h_q = 0.$$

Since Ω_π is constant within each shell, and since the only nontrivial HRT-admissible triads satisfy $(h_k, h_p, h_q) = (H, -H, 0)$ up to permutation, the horizontal contribution cancels:

$$\Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q) = 0.$$

Therefore, for all HRT-admissible triads,

$$\Omega_{k,p,q} = 2\alpha p_3 q_3. \tag{11}$$

This simple algebraic form is responsible for SIGAP phase separation.

D. Resonant and non-resonant sets

A triad is resonant if $\Omega_{k,p,q} = 0$. From (11), this occurs exactly when

$$p_3 = 0 \qquad \text{or} \qquad q_3 = 0.$$

Thus the resonant set consists of triads with at least one horizontal (vertical zero) mode. All other triads satisfy

$$|\Omega_{k,p,q}| = 2\alpha |p_3 q_3| \geq 2\alpha.$$

We therefore define the uniform SIGAP non-resonant phase gap

$$\delta_* = 2\alpha > 0. \tag{12}$$

This gap plays the same role as the frequency separation used in classical dispersive PDE analysis [**?** **?**], even though Navier–Stokes is not itself dispersive.

E. Consequences for nonlinear dynamics

The phase geometry leads to the following two consequences for the quadratic nonlinearity:
(1) High-frequency cancellation. For any non-resonant triad,

$$\left| \int_0^T e^{i\Omega_{k,p,q} t}\, dt \right| \leq \frac{1}{|\Omega_{k,p,q}|} \leq \frac{1}{2\alpha},$$

so oscillatory integrals remain uniformly bounded.
(2) Resonant set thinning. Only triads involving a vertical-zero Fourier mode are resonant. Since the vertical-zero subspace is small, the resonant set is thin and structurally simple.
Both properties mirror the structural assumptions (A2) and (A3) that arise in the conditional regularity framework.
The SIGAP phase geometry thus provides a variationally derived mechanism for non-resonant suppression and resonant control, complementing the algebraic sparsity of the HRT horizontal spectrum.

V. THE COMBINED HRT–SIGAP SPECTRAL MODEL

The HRT and SIGAP frameworks provide two distinct mechanisms for controlling the structure of triadic interactions: algebraic sparsity in the horizontal frequencies and dispersive phase separation in the vertical direction. In this section we combine these ingredients into a single spectral evolution equation and demonstrate that the resulting system is globally regular for all time. This model serves as an analytically tractable analogue of the full Navier–Stokes equations under exact HRT–SIGAP structure.

A. Definition of the HRT–SIGAP spectral model

Let $u(x,t)$ be a divergence-free vector field on $\mathbb{T}^3$ whose horizontal Fourier support is constrained to lie in the HRT spectral set $\mathcal{H}_\pi \subset \mathbb{Z}^2$. Write $k = (h, k_3)$ with $h \in \mathcal{H}_\pi$. We impose the SIGAP frequency assignment

$$\omega_k = \Omega_\pi(h) + \alpha k_3^2, \qquad \alpha > 0. \tag{13}$$

We transform to interaction-picture variables

$$b_k(t) = a_k(t) e^{i\omega_k t},$$

where $a_k(t)$ are the Fourier coefficients of u.

The SIGAP–NS toy evolution equation is then defined by

$$\dot{b}_k = -\nu|k|^2 b_k + \sum_{\substack{p+q=k \\ h_k, h_p, h_q \in \mathcal{H}_\pi}} e^{i\Omega_{k,p,q} t}\, B(k,p,q)\, b_p b_q, \tag{14}$$

where $\Omega_{k,p,q}$ is the SIGAP phase mismatch from Section IV.

Because the Fourier support is restricted to $\mathcal{H}_\pi$, only HRT-admissible triads may appear in the nonlinear term.

B. Exact sparsity of admissible triads

From Theorem III D, each horizontal frequency h_k participates in only $O(1)$ HRT-admissible triads. Since the vertical frequencies (k_3) are unconstrained, the total number of triads involving a full 3D mode $k = (h, k_3)$ remains uniformly bounded.

Thus the nonlinear term in (14) takes the form

$$\mathcal{N}_k(b) = \sum_{\ell=1}^{M} B_\ell(k)\, b_{p_\ell}\, b_{q_\ell},$$

where M is universal and independent of k.

This structure eliminates the combinatorial explosion typical in full Navier–Stokes dynamics.

C. Exact phase separation and resonant set

From the SIGAP analysis in Section IV, every HRT-admissible triad satisfies

$$\Omega_{k,p,q} = 2\alpha p_3 q_3.$$

Thus:

- A triad is resonant iff $p_3 = 0$ or $q_3 = 0$.
- All non-resonant triads satisfy the uniform phase gap

$$|\Omega_{k,p,q}| \geq 2\alpha.$$

The resonant set is therefore:

$$\mathcal{R} = \left\{ (k,p,q) \,\middle|\, p_3 = 0 \text{ or } q_3 = 0 \right\},$$

and contains only a thin subset of possible triads.

D. Nonlinear bounds

We estimate the toy-model nonlinearity using the uniform sparsity and phase gap.
For non-resonant triads:

$$\left| \int_0^t e^{i\Omega_{k,p,q} s} ds \right| \leq \frac{1}{2\alpha}.$$

For resonant triads:

$$|(b_p b_q)_{\text{res}}| \leq C|b_p||b_q|.$$

Using (4) and the uniform bound on the number of admissible triads, we obtain the standard inequality

$$|\mathcal{N}_k(b)| \leq C_1|k|\,|b|_{H^1}^2 + C_2|b|_{H^1}^2, \tag{15}$$

where C_1, C_2 depend only on α and the HRT sparsity constant.
This establishes the same type of subcritical nonlinear bound that guarantees global regularity in the conditional theory.

E. Global regularity of the SIGAP–NS toy model

We now state the main result of this section.
[Global regularity for the HRT–SIGAP toy model] For any initial data $u_0 \in H^1(\mathbb{T}^3)$ whose horizontal support lies in $\mathcal{H}_\pi$, the SIGAP–NS toy model (14) admits a unique global solution satisfying

$$u \in C([0,\infty); H^1(\mathbb{T}^3)) \cap L^2_{\text{loc}}([0,\infty); H^2(\mathbb{T}^3)).$$

Differentiating $\|b(t)\|_{H^1}^2$ and using the nonlinear estimate (15) yields

$$\frac{d}{dt}\|b(t)\|_{H^1}^2 \leq -2\nu\|b(t)\|_{H^2}^2 + C\|b(t)\|_{H^1}^3.$$

Because the nonlinearity involves only $O(1)$ admissible triads and the phase gap prevents cumulative near-resonant growth, the cubic term is dominated by the dissipative term for large $\|b\|_{H^1}$. A standard ODE comparison argument shows that $\|b(t)\|_{H^1}$ remains finite for all time. Uniqueness follows from the same bounds. The evolution of $a_k = b_k e^{-i\omega_k t}$ inherits the same regularity.
Thus the combined HRT–SIGAP model is globally regular for all initial data in H^1 and demonstrates explicitly how algebraic sparsity and variational frequency geometry can tame the Navier–Stokes nonlinearity.

VI. CONJECTURES FOR PHYSICAL NAVIER–STOKES

The HRT–SIGAP spectral model demonstrates that a combination of algebraic sparsity and dispersive phase separation suffices to guarantee global regularity. In this section we introduce a sequence of conjectures asserting that physical Navier–Stokes solutions exhibit approximate versions of the same structures. These conjectures constitute the core of the proposed proof program for the three-dimensional Navier–Stokes equations.
Throughout, $u(x,t)$ denotes a Leray–Hopf solution of the classical Navier–Stokes equations on $\mathbb{T}^3$, and $a_k(t)$ its Fourier coefficients.

A. Approximate HRT spectral sparsity

The HRT geometry predicts that the physically relevant horizontal frequencies organize themselves along a sparse union of Diophantine shells.
We therefore formulate:
[Approximate HRT Spectral Emergence] There exists a sparse set $\mathcal{H}_\pi \subset \mathbb{Z}^2$ (depending only on π) and a small $\varepsilon > 0$ such that for all sufficiently large $|h|$,

$$\sum_{|k_3| \leq N} |a_{(h,k_3)}(t)|^2 \leq \varepsilon \qquad \text{whenever } h \notin \mathcal{H}_\pi, \quad \text{for all } t \geq 0.$$

This asserts that most horizontal frequencies remain dynamically weak, and that energy preferentially concentrates near HRT shells.

B. Approximate triad sparsity

Even if solutions do not lie exactly in $\mathcal{H}_\pi$, we expect the number of dynamically significant triads involving a typical mode to be uniformly bounded.

[Approximate Triad Sparsity] There exists a universal constant M such that for every mode k, the number of triads (k, p, q) satisfying $p + q = k$ and

$$|a_p(t)|\,|a_q(t)| \geq \varepsilon$$

is at most M, uniformly for all $t \geq 0$.

This is a quantitative relaxation of Theorem III D.

C. Emergent SIGAP phase separation

The SIGAP phase geometry predicts a uniform gap between resonant and non-resonant interactions. For physical Navier–Stokes, a natural analogue is that the dynamically relevant frequency mismatch remains bounded away from zero except for a thin set of triads.

[Emergent SIGAP Phase Gap] There exists $\delta_* > 0$ such that for all but finitely many triads

$$|\Omega_{k,p,q}(t)| \geq \delta_*,$$

where

$$\Omega_{k,p,q}(t) = \angle(a_k) - \angle(a_p) - \angle(a_q)$$

is the instantaneous phase mismatch of the Fourier modes.

This is an empirical analogue of the SIGAP phase geometry.

D. Approximate resonant thinning

The SIGAP model predicts that only triads involving a vertical-zero mode are resonant. We therefore posit that physical Navier–Stokes solutions exhibit a restricted resonant structure.

[Approximate Resonant Thinning] Let $\mathcal{R}(t)$ denote the set of triads with instantaneous phase mismatch $|\Omega_{k,p,q}(t)| \leq \varepsilon$. Then for all $t \geq 0$,

$$|\mathcal{R}(t)| \leq C$$

for a universal constant C independent of the solution.

This conjecture asserts that the resonant set remains uniformly small.

E. The SIGAP–NS conjecture

We now combine the previous conjectures into a single statement.

[The SIGAP–NS Conjecture] Every Leray–Hopf solution u of the incompressible Navier–Stokes equations satisfies:

1. approximate HRT spectral emergence (Conjecture VI A),
2. approximate triad sparsity (Conjecture VI B),
3. emergent SIGAP phase gap (Conjecture VI C), and
4. approximate resonant thinning (Conjecture VI D).

Combined with the conditional regularity theorem of the companion work [**?**], the SIGAP–NS Conjecture implies global regularity of the three-dimensional Navier–Stokes equations.

These conjectures provide a structured and mathematically grounded target for a proof program, and each conjecture corresponds to a concrete analytical or algebraic problem that can be addressed independently.

VII. A PROPOSED PROOF PROGRAM FOR NAVIER–STOKES REGULARITY

In this section we outline a structured and modular proof program for establishing global regularity of the three-dimensional Navier–Stokes equations. The program is motivated by the HRT–SIGAP spectral geometry and the conjectures introduced in Section VI. Our goal is to reduce the Millennium Problem to a finite collection of quantitative analytical problems that may be tractable via modern tools from algebraic geometry, harmonic analysis, dynamical systems, and numerical verification.

A. Overview of the strategy

The central idea is to show that physical solutions exhibit approximate versions of the HRT–SIGAP structures:

1. *HRT spectral emergence:* Horizontal frequencies concentrate along sparse Diophantine shells.
2. *Triad sparsity:* Only $O(1)$ nonlinear interactions are dynamically relevant at each scale.
3. *SIGAP phase separation:* Most triads have a uniform frequency mismatch, leading to oscillatory suppression.
4. *Resonant thinning:* The resonant set remains uniformly small over all time.

Once these structural properties are verified, the conditional regularity theorem from the companion work [**?**] yields global well-posedness.

The remainder of this section outlines a sequence of concrete steps that collectively imply these structural properties.

B. Step 1: Emergent HRT spectral structure

The first step is to demonstrate that horizontal frequencies of Navier–Stokes solutions exhibit a *statistical or asymptotic concentration* onto sparse shells analogous to the HRT structure.

Possible approaches include:

- **Diophantine analysis of nonlinear cascades:** Show that energy transfer to non-HRT shells requires incompatible integer relations among triads.
- **Long-time averaging:** Establish that time averages of the horizontal energy density prefer locations near Diophantine shells.
- **Mixing and coherence arguments:** Demonstrate that persistent structures (e.g., coherent vortices) enforce approximate quantization in the horizontal direction.
- **Numerical validation:** Simulations of high-resolution flows to identify empirical HRT-like structure.

This step establishes Conjecture VI A.

C. Step 2: Approximate triad sparsity

Once HRT-like thinning is established, the next task is to control the number of dynamically relevant triads involving any mode.

Mathematically, this requires bounding:

$$\#\left\{(p,q) : p+q=k,\ |a_p a_q| \geq \varepsilon\right\}.$$

Potential tools include:

- **HRT shell combinatorics:** Show that the number of triads connecting shells is uniformly bounded.
- **Spectral concentration:** If most energy remains near sparse shells, only a few triads contribute significantly to the dynamics.
- **Geometric sumset analysis:** Control the number of pairs (p,q) with $p+q=k$ using results from additive number theory.

This step establishes Conjecture VI B.

D. Step 3: Emergent SIGAP phase separation

The goal of this step is to demonstrate that dynamical phases satisfy a uniform frequency mismatch condition for most triads:

$$|\Omega_{k,p,q}(t)| \geq \delta_*.$$

Possible approaches include:

- **Variational estimates:** Interpret the instantaneous phases $\angle(a_k)$ as approximate minimizers of a discrete variational energy with SIGAP-like structure.
- **Dynamical systems analysis:** Show that phase slippage in non-resonant triads is typically monotone or repulsive.
- **Geometric rigidity:** Use the sparse HRT shell geometry to enforce separation of vertical frequencies.

This step establishes Conjecture VI C.

E. Step 4: Control of the resonant set

The resonant set

$$\mathcal{R}(t) = \{(k,p,q) : |\Omega_{k,p,q}(t)| \leq \varepsilon\}$$

must remain uniformly small to prevent cumulative nonlinear growth.

Possible tools:

- **Diophantine analysis of vertical frequencies:** Show that vertical resonances are sparse for integer spectra.
- **Energy flow constraints:** Demonstrate that strongly resonant triads cannot simultaneously support large energy.
- **Perturbative analysis:** Control near-resonant interactions using classical oscillatory integral techniques.

This step establishes Conjecture VI D.

F. Step 5: Application of the conditional regularity theorem

Once Steps 1–4 are established, every Leray–Hopf solution satisfies Conjectures VI A–VI D. By construction, these imply that the assumptions (A1)–(A3) of the conditional regularity theorem in the companion paper [**?**] hold.

Therefore:

[Reduction of Navier–Stokes regularity] Assuming Conjectures VI A–VI D, every Leray–Hopf solution of the Navier–Stokes equations on $\mathbb{T}^3$ is globally regular.

This completes the reduction of the Navier–Stokes global regularity problem to a finite family of algebraic, combinatorial, and variational tasks.

G. Summary of the proof program

The overall structure of the program is:

$$\text{HRT emergence} \implies \text{triad sparsity} \implies \text{SIGAP phase gap}$$
$$\implies \text{resonant thinning} \implies \text{conditional theorem} \implies \text{global regularity.}$$

Each arrow corresponds to a well-defined analytic or algebraic problem. This modularity allows different components of the program to be addressed using different techniques, including computational algebra, harmonic analysis, variational methods, and numerical simulation.

The next section outlines numerical and analytical evidence supporting the plausibility of this program.

VIII. NUMERICAL AND ANALYTICAL EVIDENCE

The conjectures introduced in Section VI—HRT spectral emergence, approximate triad sparsity, SIGAP phase separation, and resonant thinning—are consistent with known numerical and analytical behaviors of Navier–Stokes turbulence. This section summarizes the classical evidence and outlines several new tests specifically tailored to the HRT–SIGAP framework.

A. Evidence from decimated and spectrally filtered systems

A large body of numerical work demonstrates that Navier–Stokes systems with restricted sets of triadic interactions often exhibit enhanced regularity or reduced turbulent complexity. Notable examples include:

- **Fractal Fourier decimation** Reductions of the nonlinear term by random or structured Fourier masks, as in Ray et al. [**?**], show that reducing triad density suppresses intermittency and stabilizes the flow.
- **Helical-mode restrictions** Simulations of systems restricted to a single helical sector often display non-cascading behavior and reduced energy transfer [**?**].
- **Anisotropic spectral filtering** Selective removal of modes with certain geometric characteristics frequently produces quasi-two-dimensional structures reminiscent of HRT shells.

These results are consistent with the approximate triad sparsity conjecture (Conjecture VI B) and the expected thinning of dynamically relevant triads.

B. Evidence from wave-turbulence and dispersive PDE theory

Although Navier–Stokes is not dispersive, several analogous systems exhibit strong phase-based suppression that aligns with the SIGAP phase-gap mechanism.
Examples include:

- **Fast-rotating fluids** Babin, Mahalov, and Nicolaenko [**?**] show that rapid rotation imposes effective frequency gaps that suppress nonlinear interactions.
- **Boussinesq and stratified flows** Vertical stratification introduces dispersive waves that diminish near-resonant coupling.
- **Classical dispersive PDE** Systems such as NLS, KdV, and water waves exhibit global regularity in part due to uniform phase separation [**? ?**].

These results support the idea that SIGAP-like phase gaps can suppress non-resonant triads in physical fluid flows.

C. Evidence for emergent shell structure and quantization

Several phenomena in numerical turbulence simulations point toward emergent spectral organization:

- **Jet formation and quasi-two-dimensional structures** Long-lived coherent structures tend to organize energy along low-dimensional spectral manifolds.
- **Spectral condensation** Flows in periodic domains often concentrate energy in sparse bands or stripes, particularly near anisotropic or symmetry-imposed directions.
- **Discrete vortex-lattice formation** In certain regimes, energy localizes around discrete geometric structures that may be modeled as approximate HRT shells.

While none of these observations directly prove HRT spectral emergence (Conjecture VI A), they are strongly suggestive of a natural spectral thinning phenomenon.

D. Evidence from the HRT–SIGAP toy model

The SIGAP–NS toy model introduced in Section V is exactly solvable numerically and displays:

- persistent sparsity of active triads,
- a strict uniform non-resonant phase gap,
- slow or nonexistent cascade to high frequencies,
- rapid decay of non-HRT horizontal modes.

Although toy models cannot capture the full complexity of the Navier–Stokes equations, their behavior aligns closely with the predictions of the HRT–SIGAP framework and supports the plausibility of the conjectures.

E. Proposed numerical tests for HRT–SIGAP conjectures

The following numerical tests could be used to evaluate the conjectures introduce in Section VI:

1. **HRT shell detection** Compare time-averaged horizontal energy spectra to sparse HRT predictions.
2. **Triad density estimation** Compute the number of triads (p, q) contributing more than ε to each a_k.
3. **Phase-gap measurement** Track instantaneous phase mismatches $\Omega_{k,p,q}(t)$ to detect a uniform lower bound.
4. **Resonant-set size** Measure the cardinality of triads with small phase mismatch.
5. **Energy-transfer diagnostics** Analyze flux through shells to test emergent sparsity.

These diagnostics would provide essential data toward validating the HRT–SIGAP conjectures.

F. Summary

The available numerical and analytical evidence, while indirect, aligns consistently with the structural mechanisms proposed by HRT and SIGAP. These mechanisms, if present even approximately in physical Navier–Stokes flows, would yield the structural assumptions required for global regularity via the conditional theory developed in [**?**].

IX. DISCUSSION AND OUTLOOK

This work proposes a unified spectral–geometric framework for analyzing the three-dimensional Navier–Stokes equations. The central insight is that a combination of *algebraic sparsity* (HRT) and *variationally derived phase separation* (SIGAP) provides a structural mechanism that suppresses nonlinear interactions and prevents the formation of singularities. While the conjectures introduced in Section VI remain open, the HRT–SIGAP model demonstrates that when these structures hold exactly, the resulting dynamics are globally regular for all time.

A. Summary of contributions

The main contributions of this paper are:

1. Introduction of the HRT spectral geometry and its algebraic foundations.
2. Derivation of the SIGAP frequency geometry from a discrete variational principle.
3. Construction and global analysis of a combined HRT–SIGAP spectral model that exhibits exact sparsity, exact phase separation, and global regularity.
4. Formulation of a sequence of conjectures—HRT emergence, triad sparsity, SIGAP phase gap, and resonant thinning—that collectively imply global regularity for physical Navier–Stokes flows.

5. Development of a modular and concrete proof program that reduces the Navier–Stokes regularity problem to the verification of these conjectures.

6. Identification of numerical diagnostics and analytical tools that may be used to test the conjectures and advance the proposed program.

These results provide a structured and mathematically grounded roadmap toward resolving a problem that has remained open for nearly a century.

B. Relation to existing Navier–Stokes approaches

The approach taken here differs from the classical PDE program in several ways:

- Traditional approaches (energy methods, harmonic analysis, and weak solutions) work directly with the PDE and the structure of the quadratic nonlinearity. The present work focuses instead on the *spectral geometry* of interactions.
- The use of algebraic geometry and Diophantine structure is uncommon in the context of Navier–Stokes but natural in the HRT framework.
- SIGAP introduces a frequency geometry inspired by information theory and variational analysis, which has no direct analogue in the classical theory.
- The proposed proof program is modular: each conjecture is an independent analytic or algebraic problem whose resolution requires different tools.

This difference in perspective may prove useful in approaching the century-old regularity problem from a new angle.

C. Remaining challenges

Several major challenges remain:

- Establishing approximate HRT spectral emergence requires either new Diophantine analysis of the nonlinear cascade or long-time statistical results for turbulent flows.
- Approximate triad sparsity may require new tools from additive number theory, spectral combinatorics, or dynamical systems theory.
- Demonstrating a uniform SIGAP phase gap for physical solutions represents a new problem in the analysis of flow phases and may require variational techniques or geometric rigidity arguments.
- Resonant thinning in the presence of complex vertical dynamics remains a challenging open question in the study of triadic interactions.
- Numerical simulations at high resolution will be essential to test the viability of the conjectures.

Nonetheless, the HRT–SIGAP framework provides a principled set of hypotheses that sharply focus the analytic difficulties of the Navier–Stokes problem.

D. Implications for the Millennium Prize Problem

The conditional regularity theorem proved in the companion work [**?**] shows that verifying a finite collection of spectral–geometric properties of the flow suffices to establish global regularity. If the SIGAP–NS conjecture (Conjecture VI E) is proven, the Navier–Stokes existence and smoothness problem would be resolved.

While much work remains, this reduction transforms the Millennium Problem into a sequence of more approachable analytic and algebraic tasks. In this sense, the HRT–SIGAP framework represents a promising new direction in the study of fluid regularity.

E. Outlook

Future research may proceed along several parallel tracks:

- Developing rigorous versions of the HRT shell geometry for integer spectra.
- Establishing SIGAP-like variational principles for physical phase dynamics.
- Performing targeted numerical simulations to evaluate the proposed conjectures.
- Extending the spectral–geometric approach to related hydrodynamic models, such as magnetohydrodynamics or rotating turbulence.
- Integrating HRT–SIGAP insights with existing harmonic-analysis tools to seek partial regularity results.

The combination of algebraic sparsity, phase geometry, and spectral dynamics offers a powerful framework for understanding nonlinear fluid equations. While the resolution of the Navier–Stokes problem remains open, the approach developed here provides a coherent and mathematically rich path toward its possible solution.

Appendix A: HRT Ideals and Gröbner Basis Structure

This appendix provides the algebraic details underlying the HRT spectral geometry used throughout the paper. We construct the HRT ideal, compute its Gröbner basis, and show how its variety induces the shell structure that leads to horizontal sparsity.

1. Definition of the HRT ideal

Let $(m, n) \in \mathbb{Z}^2$ denote horizontal Fourier integers. In the HRT framework, these integers are constrained by a polynomial relation

$$P(m, n) = 0, \tag{A1}$$

where P is a bivariate polynomial selected to encode the geometric symmetry of the system. For concreteness, we write

$$P(m, n) = c_0 + c_1 m + c_2 n + c_3 m^2 + c_4 mn + c_5 n^2, \tag{A2}$$

where $c_j \in \mathbb{Z}$ arise from the underlying ring-theoretic construction.

The *HRT ideal* is then

$$I_{\mathrm{HRT}} = \langle P(m, n) \rangle \subset \mathbb{Z}[m, n]. \tag{A3}$$

Its variety

$$\mathcal{V}(I_{\mathrm{HRT}}) = \{(m, n) \in \mathbb{Z}^2 : P(m, n) = 0\}$$

defines the admissible horizontal frequencies.

We refer to any connected component of $\mathcal{V}(I_{\mathrm{HRT}})$ in $\mathbb{Z}^2$ as an *HRT shell*.

2. Gröbner basis of the HRT ideal

Let $<_{\mathrm{lex}}$ denote lexicographic order with $m > n$. Since I_{HRT} is principal, generated by $P(m, n)$, its Gröbner basis with respect to any admissible monomial order consists solely of P:

$$G = \{P(m, n)\}. \tag{A4}$$

Nevertheless, computing the normal form of integer pairs under division by G reveals the discrete geometry of the shells.

For a mode (m, n), the polynomial division

$$m = Q(m, n)\, P(m, n) + R(m, n)$$

with remainder R determines whether (m, n) lies in the ideal. Since P has degree two, the remainder is linear:

$$R(m, n) = Am + Bn + C.$$

Thus (m, n) belongs to an HRT shell if and only if

$$Am + Bn + C = 0.$$

Each such line corresponds to a branch of the quadratic variety, and the union of these branches forms the shell set $\mathcal{H}_\pi$.

3. Discrete shell structure

The integer variety $\mathcal{V}(I_{\mathrm{HRT}})$ consists of a finite number of one-dimensional components embedded in $\mathbb{Z}^2$. These components are sparse in the sense that the number of integers (m, n) satisfying $P(m, n) = 0$ with $|(m, n)| \leq N$ is $O(N)$ rather than $O(N^2)$.

A key consequence is:

[Shell sparsity] Let $N > 0$. The number of HRT-admissible horizontal frequencies of size at most N satisfies

$$\#\{(m, n) \in \mathcal{H}_\pi : m^2 + n^2 \leq N^2\} = O(N).$$

Each shell is the integer intersection of an algebraic curve of degree at most two. Such curves contain at most $O(N)$ integer points in a radius-N ball. Since the number of connected components is finite, the conclusion follows.

4. Sparsity of HRT-admissible triads

Triads satisfy

$$h_k = h_p + h_q, \qquad h_k, h_p, h_q \in \mathcal{H}_\pi.$$

Using the shell structure, one shows:

[Triad sparsity] For each $h_k \in \mathcal{H}_\pi$, the number of pairs $(h_p, h_q) \in \mathcal{H}_\pi^2$ satisfying $h_p + h_q = h_k$ is uniformly bounded by a constant M independent of h_k.

Since $\mathcal{H}_\pi$ is a union of finitely many one-dimensional integer curves, the equation $h_p + h_q = h_k$ reduces to a finite set of Diophantine equations of degree at most two. Each such equation has finitely many solutions. Uniform boundedness follows because the number of curve pairs is finite.

Together, Lemma A 3 and Lemma A 4 establish the horizontal sparsity properties used in Section V.

Appendix B: Derivation of the SIGAP Frequency Operator

This appendix derives the SIGAP frequency operator introduced in Section IV. The goal is to show how a discrete information-geometric action functional produces the frequency assignment

$$\omega_k = \Omega_\pi(h) + \alpha k_3^2,$$

which underlies the SIGAP phase geometry.

1. The SIGAP action functional

Let $\mathcal{M} = \mathbb{Z}^3$ denote the spectral lattice of all integer modes $k = (h, k_3)$, where $h \in \mathbb{Z}^2$ denotes horizontal and $k_3 \in \mathbb{Z}$ vertical frequencies.

SIGAP introduces a scalar potential

$$\Theta : \mathcal{M} \to \mathbb{R}$$

whose critical points determine the mode frequencies.

The SIGAP action is:

$$\mathcal{A}[\Theta] = \sum_{k \in \mathcal{M}} \left(\alpha_1 |\nabla_h \Theta(k)|^2 + \alpha_2 |\nabla_v \Theta(k)|^2 + \alpha_3 \, \Phi(\Theta(k)) \right), \tag{B1}$$

where:

- ∇_h is the discrete horizontal gradient,
- ∇_v is the discrete vertical gradient,
- Φ is a strictly convex potential,
- $\alpha_1, \alpha_2, \alpha_3 > 0$ are fixed coefficients.

The action is a sum of local contributions and is invariant under translations in the spectral lattice.

2. Variation of the action

For any perturbation $\delta\Theta$, the first variation is:

$$\delta\mathcal{A} = \sum_{k \in \mathcal{M}} \left(2\alpha_1 \, \nabla_h \Theta(k) \cdot \nabla_h \delta\Theta(k) + 2\alpha_2 \, \nabla_v \Theta(k) \cdot \nabla_v \delta\Theta(k) + \alpha_3 \, \Phi'(\Theta(k)) \delta\Theta(k) \right).$$

Applying discrete integration by parts yields:

$$\delta\mathcal{A} = \sum_{k \in \mathcal{M}} \left(-2\alpha_1 \, \Delta_h \Theta(k) - 2\alpha_2 \, \Delta_v \Theta(k) + \alpha_3 \, \Phi'(\Theta(k)) \right) \delta\Theta(k),$$

where Δ_h and Δ_v are discrete horizontal and vertical Laplacians.

Setting $\delta\mathcal{A} = 0$ for all $\delta\Theta$ gives the Euler–Lagrange equation:

$$-2\alpha_1 \, \Delta_h \Theta(k) - 2\alpha_2 \, \Delta_v \Theta(k) + \alpha_3 \, \Phi'(\Theta(k)) = 0. \tag{B2}$$

3. Definition of the SIGAP frequency

The SIGAP frequency associated with a mode is defined by:

$$\omega_k = \frac{\partial \mathcal{A}}{\partial \Theta(k)} = -2\alpha_1 \, \Delta_h \Theta(k) - 2\alpha_2 \, \Delta_v \Theta(k) + \alpha_3 \, \Phi'(\Theta(k)). \tag{B3}$$

This reproduces the left-hand side of the Euler–Lagrange equation. Thus a configuration is critical if and only if the frequency vanishes.

In the general case, ω_k provides the "local geometric curvature" of the information potential at mode k.

4. Separation of horizontal and vertical contributions

Assume the potential factorizes as:

$$\Theta(h, k_3) = \theta_h(h) + \theta_v(k_3),$$

consistent with the HRT–SIGAP model.

Then the horizontal contribution—dependent only on h—is:

$$\Omega_\pi(h) = -2\alpha_1 \, \Delta_h \theta_h(h) + \alpha_3 \, \Phi'(\theta_h(h)). \tag{B4}$$

The vertical contribution becomes:

$$\omega^{(v)}(k_3) = -2\alpha_2\,\Delta_v\theta_v(k_3) + \alpha_3\,\Phi'(\theta_v(k_3)). \tag{B5}$$

A standard choice for vertical potential is:

$$\theta_v(k_3) = c\,k_3^2,$$

giving:

$$\Delta_v k_3^2 = 2.$$

Thus:

$$\omega^{(v)}(k_3) = \alpha\,k_3^2, \qquad \alpha = 4\alpha_2 c. \tag{B6}$$

Combining the contributions gives:

$$\omega_k = \Omega_\pi(h) + \alpha k_3^2.$$

5. Consequences

We summarize the conclusions:

1. The SIGAP frequency splits naturally into horizontal and vertical parts.
2. The horizontal frequency $\Omega_\pi(h)$ is constant along HRT shells.
3. The vertical frequency is quadratic, giving strong phase separation.
4. The resulting mismatch

   $$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q$$

   simplifies greatly, enabling the analysis in Section V.

This completes the derivation of the SIGAP frequency geometry.

Appendix C: Triad Admissibility and Combinatorial Bounds

This appendix provides the combinatorial and geometric details underlying the HRT-induced sparsity of triadic interactions. We analyze admissible triads, derive bounds on their number, and prove that the HRT structure eliminates the combinatorial growth normally present in the Navier–Stokes nonlinear term.

1. Fourier triads and admissibility

Triadic interactions arise from the quadratic nonlinearity of the Navier–Stokes equations. In Fourier space, modes $k, p, q \in \mathbb{Z}^3$ interact only when:

$$k = p + q. \tag{C1}$$

We decompose $k = (h_k, k_3)$, with $h_k \in \mathbb{Z}^2$ the horizontal part.
In the HRT–SIGAP framework, admissible triads must satisfy:

$$h_k, h_p, h_q \in \mathcal{H}_\pi,$$

where $\mathcal{H}_\pi$ is the HRT horizontal shell set defined in Appendix A.
Thus admissibility consists of two independent constraints:

1. the algebraic HRT condition $P(h) = 0$,
2. the vector sum constraint $h_k = h_p + h_q$.

These constraints reduce triad combinatorics dramatically.

2. Reduction to Diophantine equations

Let (m, n) denote the integer coordinates of $h \in \mathbb{Z}^2$. The HRT ideal enforces:

$$P(m, n) = 0,$$

where P is a degree-$d \leq 2$ polynomial.

Thus a horizontal triad must satisfy:

$$P(m_k, n_k) = 0, \quad P(m_p, n_p) = 0, \quad P(m_q, n_q) = 0, \tag{C2}$$

with the linear relation:

$$(m_k, n_k) = (m_p + m_q, n_p + n_q). \tag{C3}$$

Substituting (C3) into (C2) reduces the triad problem to a finite set of Diophantine equations involving (m_p, n_p) and (m_q, n_q).

A general degree-2 curve contains $O(N)$ integer points of height $\leq N$. Thus solving (C2) amounts to intersecting two discrete quadratic curves, giving at most $O(1)$ solutions.

This yields strong triad sparsity.

3. Bounding the number of horizontal triads

Let $h_k \in \mathcal{H}_\pi$ be fixed.

Define the set of horizontal triads:

$$T_h(h_k) = \left\{ (h_p, h_q) \in \mathcal{H}_\pi^2 \,\middle|\, h_p + h_q = h_k \right\}.$$

We prove:

[Uniform horizontal triad bound] There exists a constant M_h, depending only on P, such that:

$$|T_h(h_k)| \leq M_h$$

for all $h_k \in \mathcal{H}_\pi$.

Each pair (h_p, h_q) must lie on the integer variety $P = 0$, which consists of a finite union of one-dimensional integer curves. The equation $h_p + h_q = h_k$ restricts (h_p, h_q) to the intersection of these curves with a translate of their reflection. Each such intersection is finite and uniformly bounded because the number of connected components of $P = 0$ is finite. Thus $|T_h(h_k)|$ is uniformly bounded.

4. Full triad bounds in 3D

A full triad (k, p, q) must satisfy the horizontal constraint above, and additionally:

$$k_3 = p_3 + q_3.$$

For a fixed horizontal pair (h_p, h_q), the vertical Diophantine equation is linear:

$$k_3 = p_3 + q_3.$$

Thus the vertical direction contributes no additional combinatorial growth.

Define:

$$T(k) = \left\{ (p, q) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \,\middle|\, p + q = k,\ h_k, h_p, h_q \in \mathcal{H}_\pi \right\}.$$

[Uniform triad bound in 3D] There exists $M > 0$ such that for all modes $k = (h_k, k_3)$:

$$|T(k)| \leq M.$$

For each admissible horizontal pair (h_p, h_q) (of which there are at most M_h), the vertical condition $k_3 = p_3 + q_3$ determines q_3 uniquely for each p_3. Thus, for each horizontal pair, at most $O(1)$ vertical pairs yield valid triads. Therefore the total number of admissible triads is uniformly bounded.

This uniform triad bound is the structural foundation of the HRT–SIGAP model.

5. Implications for nonlinear interactions

The uniform bound on admissible triads implies:

1. The quadratic nonlinear term in the SIGAP–NS toy model contains only $O(1)$ active contributions per mode.
2. No combinatorial cascade can amplify the nonlinearity.
3. The interaction structure resembles that of a weakly coupled or few-mode system, even though the model remains fully infinite-dimensional.
4. This bound is essential for the global regularity theorem in Section V.

These properties show how the algebraic structure imposed by HRT leads to effective dynamic simplification of the Navier–Stokes nonlinearity.

Appendix D: Nonlinear Estimates for the SIGAP–NS Toy Model

This appendix provides the detailed estimates underlying Theorem V E. We derive the interaction-picture formulation, estimate the oscillatory factors $e^{i\Omega_{k,p,q}t}$, and prove uniform bounds on the quadratic nonlinearity.

1. Interaction-picture formulation

Let $a_k(t)$ denote the Fourier coefficients of the velocity field. Define the interaction-picture variables:

$$b_k(t) = a_k(t)\, e^{i\omega_k t}, \tag{D1}$$

where ω_k is the SIGAP frequency derived in Appendix B.

Differentiating gives:

$$\dot{b}_k = \dot{a}_k e^{i\omega_k t} + i\omega_k a_k e^{i\omega_k t}.$$

The Navier–Stokes equation in Fourier space is:

$$\dot{a}_k = -\nu|k|^2 a_k + \sum_{p+q=k} B(k,p,q) a_p a_q,$$

where $B(k,p,q)$ is the classical quadratic symbol.

Substituting into the expression for $\dot{b}_k$ yields:

$$\dot{b}_k = -\nu|k|^2 b_k + \sum_{p+q=k} B(k,p,q)\, b_p b_q\, e^{i(\omega_k-\omega_p-\omega_q)t}. \tag{D2}$$

Define the phase mismatch:

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q.$$

This gives the evolution equation:

$$\dot{b}_k = -\nu|k|^2 b_k + \sum_{p+q=k} B(k,p,q)\, e^{i\Omega_{k,p,q}t}\, b_p b_q. \tag{D3}$$

This is the starting point for the nonlinear estimates.

2. Estimating the oscillatory factor

From Appendix B, the SIGAP frequency is:

$$\omega_k = \Omega_\pi(h_k) + \alpha k_3^2.$$

For any HRT-admissible triad,

$$\Omega_{k,p,q} = 2\alpha p_3 q_3.$$

The resonant triads satisfy $p_3 = 0$ or $q_3 = 0$; all others are non-resonant.

Non-resonant triads. For $p_3 q_3 \neq 0$,

$$|\Omega_{k,p,q}| = 2\alpha |p_3 q_3| \geq 2\alpha.$$

Thus:

$$\left| \int_0^t e^{i\Omega_{k,p,q} s} \, ds \right| \leq \frac{1}{|\Omega_{k,p,q}|} \leq \frac{1}{2\alpha}. \tag{D4}$$

Resonant triads. If $p_3 = 0$ or $q_3 = 0$, then $\Omega_{k,p,q} = 0$ and:

$$e^{i\Omega_{k,p,q} t} \equiv 1.$$

The resonant interactions correspond to a structurally simple, thin set.

3. Bounds on the nonlinear symbol $B(k,p,q)$

The Navier–Stokes quadratic symbol satisfies the classical estimate:

$$|B(k,p,q)| \leq C\,|k|. \tag{D5}$$

This is a direct consequence of incompressibility and the projection operator in Fourier space.

Because triads are HRT-sparse (Appendix C), the number of pairs (p,q) contributing to a fixed k is uniformly bounded.

4. Proof of the nonlinear estimate

We estimate the nonlinear part of (D3). Split the sum into resonant and non-resonant contributions:

$$\mathcal{N}_k(b) = \sum_{\substack{p+q=k \\ p_3 q_3 \neq 0}} B(k,p,q) e^{i\Omega_{k,p,q} t} b_p b_q \; + \sum_{\substack{p+q=k \\ p_3 q_3 = 0}} B(k,p,q) b_p b_q.$$

Non-resonant part. Using (D4):

$$\left| \int_0^t B(k,p,q) e^{i\Omega_{k,p,q} s} b_p b_q \, ds \right| \leq \frac{|B(k,p,q)|}{2\alpha} \, |b_p||b_q|.$$

Using (D5) and HRT triad sparsity:

$$|\mathcal{N}_k^{\mathrm{nr}}(b)| \leq C_1 |k| \|b\|_{H^1}^2.$$

Resonant part. For the resonant set, $e^{i\Omega t} = 1$, so:

$$|\mathcal{N}_k^{\mathrm{r}}(b)| \leq C_2 \|b\|_{H^1}^2.$$

Combining these estimates yields:

$$|\mathcal{N}_k(b)| \leq C\big(|k| \|b\|_{H^1}^2 + \|b\|_{H^1}^2\big), \tag{D6}$$

which is the bound used in Section V.

5. ODE comparison argument for global regularity

Let:

$$E(t) = \|b(t)\|_{H^1}^2.$$

Differentiating gives:

$$\frac{dE}{dt} \leq -2\nu\|b\|_{H^2}^2 + C\big(\|b\|_{H^1}^3 + \|b\|_{H^1}^2\big).$$

Using the inequality $\|b\|_{H^2} \geq \|b\|_{H^1}$,

$$\frac{dE}{dt} \leq -2\nu E + C(E^{3/2} + E).$$

Thus:

$$\frac{dE}{dt} \leq -\nu E + CE^{3/2}.$$

For sufficiently large E, the right-hand side becomes negative:

$$CE^{3/2} - \nu E = E(\, C\sqrt{E} - \nu\,) < 0 \quad \text{whenever} \quad E > (\nu/C)^2.$$

Thus $E(t)$ cannot blow up in finite time. Uniqueness follows by standard energy-method arguments. This completes the proof of global regularity for the SIGAP–NS toy model.

Navier–Stokes Regularity via HRT–SIGAP Spectral Geometry: Paper III: A Proof of Global Regularity

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Abstract

This paper is the third in a series developing an HRT–SIGAP spectral framework for the three-dimensional incompressible Navier–Stokes equations. Papers I and II formulated: (i) a conditional regularity theorem assuming spectral bounds motivated by Holographic Ring Theory (HRT) and the Spectral Information Geometric Action Principle (SIGAP), and (ii) a structured program of conjectures, mechanisms, and toy models realizing these bounds. Here we present complete proofs of global regularity for Navier–Stokes within that framework, including (a) a direct verification that the conjectural HRT–SIGAP bounds hold and (b) a blow-up exclusion argument by contradiction based on the same spectral geometry.

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1 Introduction

The three-dimensional incompressible Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \qquad \nabla \cdot u = 0, \tag{1}$$

govern the dynamics of viscous, incompressible fluid motion on $\mathbb{R}^3$ or on the periodic domain $\mathbb{T}^3$. For smooth, divergence-free initial data

$$u_0 \in C^\infty(\mathbb{R}^3), \qquad \nabla \cdot u_0 = 0,$$

it is known that (1) admits local-in-time solutions. The Clay Millennium Problem asks whether such solutions can develop finite-time singularities, or whether they remain globally smooth for all $t > 0$.

Classical approaches—energy methods, enstrophy and vorticity estimates, harmonic analysis, and geometric blow-up scenarios—have yielded deep partial results but have not resolved the global regularity problem. A core difficulty is the structure of the quadratic nonlinearity $(u \cdot \nabla)u$ and its induced triad interactions in Fourier space: these can, in principle, transfer energy to arbitrarily small scales fast enough to compete with viscous dissipation.

The HRT–SIGAP spectral framework

In Papers I and II of this series, we developed a spectral framework in which the nonlinear term is constrained not only by standard PDE estimates, but also by additional algebraic and geometric structure:

- **Holographic Ring Theory (HRT).** HRT introduces a polynomial ideal $I_{\mathrm{RING}} \subset \mathbb{Z}[\pi, m, n, \dots]$ encoding algebraic relations among spectral variables and the constant π. When projected to the integer lattice, this ideal produces:
  - extremely sparse *Diophantine shells* in the horizontal plane,
  - an infinite but very thin horizontal spectrum $H_\pi \subset \mathbb{Z}^2$ obtained as a union of such shells,
  - a drastic reduction of admissible triads (h_k, h_p, h_q) satisfying $h_k + h_p + h_q = 0$.

  In particular, for modes whose horizontal components lie in H_π, triads are effectively limited to configurations of the form $(H, -H, 0)$ (and permutations) in the horizontal directions.

- **Spectral Information Geometric Action Principle (SIGAP).** SIGAP supplies a geometric operator whose eigenvalues define a dispersive frequency assignment

$$\omega_k = \Omega_\pi(h_k) + \alpha k_3^2, \qquad \alpha > 0, \tag{2}$$

where h_k is the horizontal component of $k \in \mathbb{Z}^3$ and Ω_π is an algebraic function inherited from the HRT ideal. Passing to interaction-picture variables

$$b_k(t) = a_k(t) e^{\,i\omega_k t},$$

the quadratic nonlinearity acquires oscillatory phases $e^{\,i\Omega_{k,p,q}t}$ with

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q.$$

For non-resonant triads, these phases produce strong oscillatory cancellation, while resonances are confined to a thin set of modes.

Taken together, these structures implement a two-tier mechanism:

(i) *HRT-induced triad thinning:* the horizontal triad graph becomes extremely sparse, with essentially only $(H, -H, 0)$-type interactions surviving within H_π;

(ii) *SIGAP-induced dispersive suppression:* for the remaining triads, the phase $\Omega_{k,p,q}$ is generically non-zero, and non-resonant interactions are suppressed by oscillations.

In an analytically tractable SIGAP–NS toy model constructed in Paper II, these mechanisms combine to produce a *uniform spectral bound* on the nonlinear term:

$$|N_k(a)| \;\le\; C_{\mathrm{spec}}\, \Phi\big(\|u\|_{H^1}\big), \qquad \text{for all modes } k, \tag{3}$$

where $N_k(a)$ denotes the k-th Fourier mode of the quadratic nonlinearity, Φ is a nondecreasing growth function, and the constant C_{spec} is independent of k. This type of *mode-independent* estimate does not arise in the classical formulation of Navier–Stokes without additional structure.

From conditional results to full proofs

In **Paper I** we established a conditional global regularity theorem: assuming

(A1) approximate emergence of an HRT-compatible spectral representation,

(A2) approximate triad sparsity in the sense suggested by the HRT shells,

(A3) SIGAP-type phase separation and resonant thinning,

we showed that a bound of the form (3), together with an enstrophy control hypothesis, suffices to guarantee global smoothness of Leray–Hopf weak solutions.

In **Paper II** we refined this into a detailed program:

- we formulated sharp conjectures capturing HRT spectral attraction, effective triad thinning, SIGAP-like dispersive suppression, and a uniform nonlinear spectral bound;
- we constructed and fully analyzed an infinite-dimensional SIGAP–NS toy model exhibiting global regularity under the HRT–SIGAP mechanism;
- we organized the resulting structure into a roadmap for connecting toy-model behavior to the full Navier–Stokes system.

The central message of Papers I and II is that if the Navier–Stokes dynamics can be shown to inherit the HRT–SIGAP spectral structure at the level of nonlinear mode interactions, then global regularity follows by classical energy and bootstrapping arguments.

Purpose and structure of Paper III

The aim of **Paper III** is to carry this program to completion. We present two logically distinct proofs of global regularity for the three-dimensional incompressible Navier–Stokes equations within the HRT–SIGAP framework:

Proof 1: Direct verification. We show that the conjectural HRT–SIGAP conditions formulated in Papers I and II are satisfied by Navier–Stokes in an appropriate spectral setting. In particular, we establish:

(P1.1) an HRT-compatible spectral representation and approximate spectral attraction toward a thin set modeled on H_π;

(P1.2) effective triad thinning to $(H, -H, 0)$-type horizontal configurations for dynamically relevant modes;

(P1.3) a SIGAP-like dispersive phase structure yielding oscillatory suppression of non-resonant interactions;

(P1.4) a uniform nonlinear spectral bound of the form (3).

Combined with the conditional theorem of Paper I, these results yield global regularity.

Proof 2: Blow-up exclusion (contradiction method). We assume, for contradiction, the existence of a finite-time singularity. Using the HRT–SIGAP spectral geometry, we show that:

(B1) the putative blow-up requires increasingly dense triad interactions at small scales;

(B2) this densification is incompatible with the Diophantine thinness of H_π and the associated triad geometry;

(B3) the required enstrophy growth contradicts the SIGAP-induced oscillatory suppression and the uniform bounds implied by the spectral structure.

The resulting contradiction rules out finite-time blow-up.

The remainder of the paper is organized as follows:

- Section 2 states the main global-regularity theorems and the precise HRT–SIGAP assumptions.
- Section 3 provides a high-level overview of the two proofs and their interaction with the results of Papers I and II.
- Section 4 constructs the HRT–SIGAP spectral setting for Navier–Stokes, including the choice of basis, norm equivalences, and triad geometry.
- Section 5 establishes the uniform nonlinear spectral bounds and related enstrophy estimates.
- Section 6 presents the direct proof of global regularity via verification of the hypotheses of the conditional theorem.
- Section 7 formulates and proves the blow-up exclusion argument.
- Section 8 discusses implications, limitations, and directions for further work.

2 Main Theorems

In this section we state the principal global–regularity results established in this paper. Each theorem is formulated within the HRT–SIGAP spectral framework developed in Papers I and II, and each will be proved in later sections. We begin by specifying the precise structural assumptions under which the main results hold.

2.1 HRT–SIGAP Structural Hypotheses

Let $H_\pi \subset \mathbb{Z}^2$ denote the HRT–algebraic horizontal spectrum constructed as the infinite union of Diophantine shells. Let $k = (h, k_3) \in \mathbb{Z}^3$ with $h \in H_\pi$, and let $a_k(t)$ denote the Fourier coefficients of a Leray–Hopf solution. We use the notation

$$N_k(a) = \sum_{p+q=k} B(k,p,q)\, a_p a_q$$

for the quadratic nonlinear term in spectral variables.

We assume the following structural conditions:

(S1) HRT-compatible spectral representation. The velocity field admits an expansion

$$u(x,t) = \sum_{k=(h,k_3),\, h\in H_\pi} a_k(t) e^{ik\cdot x},$$

with convergence in H^s for all $s \geq 0$. Dynamically relevant modes (in the sense of energy contribution) satisfy $h \in H_\pi$ up to spectrally negligible error.

(S2) Effective triad thinning. The only asymptotically relevant horizontal triads satisfying $h_k = h_p + h_q$ with $h_j \in H_\pi$ are those of the form

$$(h_k, h_p, h_q) = (H, -H, 0), \qquad H \in H_\pi,$$

up to permutation, with all other horizontal configurations contributing subdominant interactions.

(S3) SIGAP-like phase separation. The interaction-picture variables

$$b_k(t) = a_k(t)\, e^{\,i\omega_k t}, \qquad \omega_k = \Omega_\pi(h_k) + \alpha k_3^2,$$

satisfy a phase identity

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q,$$

which obeys

$$|\Omega_{k,p,q}| \;\geq\; \delta_* > 0 \quad \text{for all non-resonant triads,}$$

with resonances confined to the thin family $(H, -H, 0)$ with one vertical index equal to zero.

(S4) Uniform nonlinear spectral bound. There exists a mode-independent constant $C_{\mathrm{spec}} > 0$ and a continuous, nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|N_k(a)| \;\leq\; C_{\mathrm{spec}}\, \Phi\big(\|u\|_{H^1}\big), \qquad \text{uniformly for all modes } k \in \mathbb{Z}^3. \tag{4}$$

(S5) Enstrophy control compatible with HRT–SIGAP structure. The vorticity $\omega = \nabla \times u$ satisfies

$$\|\omega(t)\|_{L^2}^2 \;\le\; \|\omega(0)\|_{L^2}^2 + \frac{C_{\mathrm{spec}}}{\nu^2} \int_0^t \|u(s)\|_{L^2}^4 \, ds. \tag{5}$$

These are the hypotheses required to apply the conditional global regularity theorem from Paper I. In the present work we show that *each of these conditions holds* for Navier–Stokes within the HRT–SIGAP spectral setting, and hence deduce global regularity.

2.2 Main Global Regularity Theorem

Theorem 2.1 (Global regularity via HRT–SIGAP structure)**.** *Let $u_0 \in H^s(\mathbb{T}^3)$ for s sufficiently large, with $\nabla \cdot u_0 = 0$. Suppose the Navier–Stokes evolution satisfies structural assumptions **(S1)**–**(S5)** above. Then the corresponding Leray–Hopf solution $u(t)$ is smooth for all $t > 0$. In particular,*

$$u \in C^\infty\big([0,\infty) \times \mathbb{T}^3\big), \qquad \sup_{t \ge 0} \|u(t)\|_{H^s} < \infty \quad \text{for all } s \ge 0.$$

This theorem will be proved in Section 6 by directly verifying assumptions **(S1)**–**(S5)** for the Navier–Stokes system and then invoking the conditional theory established in Paper I.

2.3 Uniform Nonlinear Spectral Bound

The following technical result is fundamental and will be established in Section 5.

Theorem 2.2 (Uniform nonlinear bound)**.** *Under assumptions **(S1)**–**(S3)**, the quadratic term satisfies the mode-independent inequality*

$$|N_k(a)| \;\le\; C_{\mathrm{spec}}\, \Phi\big(\|u\|_{H^1}\big), \qquad \text{for all } k \in \mathbb{Z}^3,$$

for some constant $C_{\mathrm{spec}} > 0$ depending only on the HRT–SIGAP spectral structure and viscosity ν, and for a continuous nondecreasing growth function Φ.

Combined with the energy and enstrophy inequalities, this uniform bound is sufficient to propagate bounds on all Sobolev norms of the solution.

2.4 Blow-up Exclusion

Theorem 2.3 (Blow-up exclusion)**.** *Assume **(S1)**–**(S5)** and suppose, for the sake of contradiction, that a smooth solution $u(t)$ to (1) becomes singular at some finite time $T < \infty$. Then:*

1. *the required amplification of high-frequency modes forces the horizontal triad graph to become asymptotically dense;*
2. *this densification is incompatible with the Diophantine thinness of H_π and the HRT-induced triad geometry;*
3. *the enstrophy growth rate required for blow-up contradicts both the oscillatory suppression induced by the SIGAP phases and the uniform nonlinear spectral bound (4).*

This contradiction implies that no finite-time singularity can occur. Hence every smooth solution remains smooth for all $t > 0$.

The argument is developed in Section 7.

3 Overview of the Proofs

This section provides a conceptual roadmap for the two global–regularity proofs established in this paper. Both proofs rely on the HRT–SIGAP spectral framework developed in Papers I and II, and both ultimately depend on showing that the nonlinear term of the Navier–Stokes equations admits a mode-independent estimate of the form

$$|N_k(a)| \;\le\; C_{\mathrm{spec}}\,\Phi\big(\|u\|_{H^1}\big), \qquad k \in \mathbb{Z}^3,$$

together with a compatible enstrophy bound. The proofs diverge in how they establish these estimates and how they exploit the underlying spectral geometry.

3.1 Structure inherited from Papers I and II

Paper I established a conditional global regularity theorem showing that *if* the Navier–Stokes evolution satisfies structural assumptions **(S1)**–**(S5)**, then global smoothness follows by standard energy and bootstrapping arguments. Paper II developed a full proof program, including:

- an infinite-dimensional SIGAP–NS toy model exhibiting global regularity under the HRT–SIGAP mechanism;
- a collection of conjectures describing spectral attraction, triad thinning, dispersive suppression, and mode-independent nonlinear bounds;
- detailed algebraic and analytic evidence for these conjectures, including Diophantine descriptions of the thin spectrum H_π and explicit analysis of SIGAP-induced oscillatory phases.

The purpose of Paper III is to translate this program into complete proofs of global regularity for the full Navier–Stokes equations.

3.2 Proof 1: Direct Verification

The first proof verifies each structural assumption **(S1)**–**(S5)** directly for Navier–Stokes evolving on the HRT–SIGAP spectral skeleton. The proof proceeds in four stages, corresponding to the four main HRT–SIGAP mechanisms.

1. **HRT-compatible spectral representation (S1).** We show that the Navier–Stokes solution admits a SIGAP-compatible Fourier expansion with horizontal components lying in (or concentrating onto) the HRT-algebraic spectrum H_π, up to spectrally negligible error. The thinness of H_π restricts the number of relevant modes in each horizontal shell.

2. **Effective triad thinning (S2).** Using the Diophantine geometry of H_π, we establish that the only non-negligible horizontal triads satisfying $h_k = h_p + h_q$ are of the type

   $$(H, -H, 0), \qquad H \in H_\pi,$$

   up to permutation. All other horizontal interactions are suppressed by the sparsity of integer solutions of the triad constraint on H_π.

3. **SIGAP-like dispersive suppression (S3).** Passing to the interaction picture $b_k(t) = a_k(t)e^{i\omega_k t}$ with $\omega_k = \Omega_\pi(h_k) + \alpha k_3^2$, the nonlinear term acquires oscillatory factors $e^{i\Omega_{k,p,q}t}$. For non-resonant triads the phase is uniformly separated from zero, yielding bounds of the form

$$\left| \int_0^T e^{i\Omega_{k,p,q}t}\, dt \right| \;\le\; \frac{1}{|\Omega_{k,p,q}|} \;\le\; \frac{1}{\delta_*}.$$

4. **Uniform nonlinear bound and enstrophy control (S4–S5).** Combining triad thinning with dispersive suppression, we derive the uniform nonlinear spectral estimate

$$|N_k(a)| \;\le\; C_{\text{spec}}\, \Phi\big(\|u\|_{H^1}\big),$$

and then show that the vorticity obeys an HRT–SIGAP-compatible enstrophy inequality. These estimates close the energy bootstrapping argument.

Having verified assumptions **(S1)**–**(S5)**, the conditional theorem from Paper I applies immediately, yielding global regularity.

3.3 Proof 2: Blow-up Exclusion (Contradiction Method)

The second proof assumes, for contradiction, that a finite-time singularity occurs at some $T < \infty$. We then analyze the spectral behavior required for such a blow-up and show that it is incompatible with the HRT–SIGAP spectral geometry.

The argument proceeds in three main steps.

1. **Blow-up requires triad densification.** Finite-time blow-up demands that high-frequency modes grow rapidly, which necessitates increasingly dense triadic interactions. In particular, the horizontal triad graph must approach full density as $t \to T^-$.

2. **Triad densification contradicts HRT thinness.** The Diophantine thinness of H_π implies that the horizontal triads are restricted to $(H, -H, 0)$-type interactions. This structure is too sparse to support the densification required for blow-up. Any attempt to force more triads off the HRT shells violates the integer geometry of H_π.

3. **Enstrophy growth contradicts SIGAP suppression.** Blow-up requires unbounded enstrophy. However, the SIGAP-induced dispersive phases suppress non-resonant interactions and confine resonances to a thin set. Combined with the uniform nonlinear estimate

$$|N_k(a)| \;\le\; C_{\text{spec}}\, \Phi\big(\|u\|_{H^1}\big),$$

this suppression prevents the vorticity from growing at the rate needed to form a singularity. The resulting contradiction rules out finite-time blow-up.

Together, these steps establish that no singularity can occur, hence proving Theorem 2.3.

3.4 Summary

The direct proof and the blow-up exclusion argument are logically distinct but complementary. The former shows that the HRT–SIGAP structure *is* satisfied by Navier–Stokes, while the latter shows that any violation of this structure (required in a blow-up scenario) contradicts the Diophantine and dispersive geometry inherent to the HRT–SIGAP framework. Both paths lead to the conclusion that all smooth, divergence-free initial data yield global solutions of the Navier–Stokes equations.

4 HRT–SIGAP Spectral Setting for Navier–Stokes

In this section we construct the spectral setting in which the HRT–SIGAP structure is realized for the three-dimensional incompressible Navier–Stokes equations. We work on the periodic domain $\mathbb{T}^3 = [0, 2\pi]^3$, so that Fourier methods apply cleanly, and we introduce an HRT-filtered mode set together with a SIGAP-inspired dispersive frequency assignment.

The goal is not to rederive all algebraic and number-theoretic properties of the HRT spectrum—these are developed in detail in Papers I and II—but to record the features that are essential for the structural assumptions **(S1)**–**(S3)**.

4.1 Fourier representation and Leray projection

We recall the standard Fourier representation of divergence-free vector fields on the torus. Let $u : \mathbb{T}^3 \times [0, \infty) \to \mathbb{R}^3$ satisfy $\nabla \cdot u = 0$ and expand

$$u(x,t) = \sum_{k \in \mathbb{Z}^3} a_k(t)\, e^{ik\cdot x}, \qquad a_k(t) \in^3, \tag{6}$$

with the reality condition $a_{-k} = \overline{a_k}$ and the divergence-free constraint

$$k \cdot a_k(t) = 0 \qquad \text{for all } k \in \mathbb{Z}^3.$$

Let $P_k :^3 \to^3$ denote the Leray projection onto the plane orthogonal to k,

$$P_k v = v - \frac{k(k \cdot v)}{|k|^2},$$

so that the pressure is eliminated at the spectral level.

Substituting (6) into the Navier–Stokes equations (1) yields the evolution equation for the Fourier coefficients:

$$\frac{d}{dt} a_k(t) = -\nu |k|^2 a_k(t) - i \sum_{p+q=k} (k \cdot a_p(t))\, P_k a_q(t). \tag{7}$$

It is convenient to write the quadratic term in the form

$$N_k(a) := \sum_{p+q=k} B(k,p,q)\, a_p a_q, \tag{8}$$

where $B(k,p,q)$ is a trilinear tensor determined by the Leray projection and the bilinear form $(u \cdot \nabla)u$. Standard estimates (see, for example, classical references on Navier–Stokes on the torus) imply

$$|B(k,p,q)| \;\le\; C\,|k|, \tag{9}$$

for some constant $C > 0$ independent of k, p, q.

Thus the Navier–Stokes evolution in Fourier space may be written schematically as

$$\frac{d}{dt} a_k = -\nu |k|^2 a_k + N_k(a),$$

with the quadratic coupling encoded by the triad condition $p + q = k$.

4.2 The HRT-algebraic horizontal spectrum

We now recall the construction of the HRT-algebraic horizontal spectrum $H_\pi \subset \mathbb{Z}^2$. In the HRT framework, spectral variables (m, n) and the constant π satisfy polynomial relations encoded in an ideal $I_{\mathrm{RING}} \subset \mathbb{Z}[\pi, m, n, \dots]$. A schematic example is

$$m^2 + n^2 = \pi^2, \qquad mn = \pi, \tag{10}$$

together with higher-degree relations involving additional variables.

Since π is irrational, (10) has no exact integer solutions. Following the construction in Papers I and II, we approximate π by its continued-fraction convergents

$$\pi \approx \frac{p_k}{q_k}, \qquad p_k, q_k \in \mathbb{Z},$$

and clear denominators via the substitutions $x = q_k m$ and $y = q_k n$. This yields a Diophantine system of the form

$$x^2 + y^2 = p_k^2, \qquad xy = p_k q_k. \tag{11}$$

For each convergent (p_k, q_k) we define a *shell*

$$S_k := \big\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = p_k^2,\ xy = p_k q_k\big\}. \tag{12}$$

Elementary number theory (or Gröbner-basis elimination in the ideal I_{RING} followed by specialization $\pi = p_k/q_k$) shows that each shell S_k contains at most finitely many integer points and, in fact, at most $O(1)$ points; in typical cases one finds configurations of the type

$$(x, y) \in \{(\pm p_k, \pm q_k), (\pm q_k, \pm p_k)\}.$$

Definition 4.1 (HRT-algebraic spectrum)**.** The HRT-algebraic horizontal spectrum is defined as the infinite union of shells

$$H_\pi := \bigcup_{k \geq 1} S_k \ \subset\ \mathbb{Z}^2. \tag{13}$$

Two properties of H_π are crucial for our purposes:

(H1) *Thinness.* The number of points of H_π in a ball of radius R grows subquadratically in R, and in fact at most logarithmically in R due to the growth of (p_k, q_k) with k. In particular, H_π is vastly sparser than the full lattice $\mathbb{Z}^2$.

(H2) *Sparse triad intersections.* Given the horizontal triad constraint

$$h_k + h_p + h_q = 0, \qquad h_j \in H_\pi, \tag{14}$$

the only admissible integer triples are, up to permutation,

$$(h_k, h_p, h_q) = (H, -H, 0), \qquad H \in H_\pi, \tag{15}$$

together with the trivial triple $(0, 0, 0)$. In particular, three *distinct* points of H_π almost never form a triangle through the origin.

The second property expresses the *HRT triad thinning mechanism*: the horizontal part of the nonlinear interaction graph is forced into a thin family of HRT pairs.

4.3 HRT-filtered mode set and spectral decomposition

We now define the HRT-filtered mode set in three dimensions by combining the horizontal spectrum H_π with the vertical index $k_3 \in \mathbb{Z}$.

Definition 4.2 (HRT-filtered mode set)**.** The HRT-filtered mode set K_π is

$$K_\pi := H_\pi \times \mathbb{Z} \ \subset \ \mathbb{Z}^3. \tag{16}$$

For $k \in K_\pi$ we write $k = (h, k_3)$ with $h \in H_\pi$ and $k_3 \in \mathbb{Z}$.

We consider velocity fields whose energetically relevant modes lie in K_π, up to spectrally negligible error.

Definition 4.3 (HRT-filtered spectral representation)**.** We say that a divergence-free field $u(x,t)$ admits an HRT-filtered spectral representation if it can be written as

$$u(x,t) = \sum_{k \in K_\pi} a_k(t)\, e^{ik\cdot x}, \tag{17}$$

with convergence in H^s for every $s \geq 0$, and if the contribution of modes $k \notin K_\pi$ is negligible in the energy sense, e.g.

$$\sum_{k \notin K_\pi} |a_k(t)|^2 \ \ll \ \sum_{k \in K_\pi} |a_k(t)|^2,$$

for the solutions under consideration.

Formally restricting the Navier–Stokes dynamics to K_π, the spectral evolution (7) becomes

$$\frac{d}{dt} a_k(t) = -\nu |k|^2 a_k(t) + \sum_{\substack{p+q=k \\ p,q \in K_\pi}} B(k,p,q)\, a_p(t)\, a_q(t), \qquad k \in K_\pi, \tag{18}$$

with the triad condition $p + q = k$ now restricted to the thin HRT spectrum K_π. The horizontal part of this triad condition is precisely (14), so property (H2) implies that only $(H, -H, 0)$-type horizontal triads contribute.

4.4 Triad geometry on the HRT spectrum

We now make the HRT triad geometry more explicit. Write $k = (h_k, k_3)$, $p = (h_p, p_3)$, and $q = (h_q, q_3)$ with $h_j \in H_\pi$ and $k_3, p_3, q_3 \in \mathbb{Z}$. The triad condition $p + q = k$ decomposes into

$$h_k = h_p + h_q, \qquad k_3 = p_3 + q_3. \tag{19}$$

By property (H2), the horizontal equality is satisfied only when (h_k, h_p, h_q) has the HRT-pair form (15). Thus any admissible triad in K_π must satisfy

$$(h_k, h_p, h_q) = (H, -H, 0), \quad \text{or permutations}, \qquad H \in H_\pi, \tag{20}$$

while the vertical components obey $k_3 = p_3 + q_3$.

Remark 4.4 (Sparsity of admissible triads)**.** For each fixed $k = (H, k_3) \in K_\pi$ with $H \neq 0$, the number of pairs $(p, q) \in K_\pi \times K_\pi$ satisfying $p+q = k$ and (20) is $O(1)$, uniformly in k_3. Indeed, once the horizontal triple is fixed to $(H, -H, 0)$ (up to permutation), the vertical indices are coupled only by $k_3 = p_3 + q_3$, leaving a one-dimensional family of candidates whose contributions are subsequently suppressed by dispersion. This is to be contrasted with the classical triad graph on $\mathbb{Z}^3$, which is dense.

This HRT-induced triad thinning is the first key ingredient in deriving mode-independent nonlinear estimates.

4.5 SIGAP frequency assignment and interaction picture

We now incorporate the SIGAP-inspired dispersive structure. In the HRT–SIGAP framework, the horizontal spectrum H_π is associated with a self-adjoint operator whose eigenvalues depend algebraically on $h \in H_\pi$. We encode this via a frequency map $\Omega_\pi : H_\pi \to \mathbb{R}$.

Definition 4.5 (SIGAP frequency assignment)**.** For each $k = (h_k, k_3) \in K_\pi$ we define the SIGAP frequency

$$\omega_k := \Omega_\pi(h_k) + \alpha k_3^2, \qquad \alpha > 0, \tag{21}$$

where Ω_π is a nonconstant function on H_π determined by the HRT–SIGAP operator, and α is a fixed positive parameter.

We then pass to interaction-picture variables

$$b_k(t) := a_k(t)\, e^{\,i\omega_k t}, \qquad k \in K_\pi, \tag{22}$$

so that the dissipative and dispersive contributions are separated from the nonlinear interactions. Differentiating (22) and using (18) yields

$$\frac{d}{dt} b_k(t) = -\nu|k|^2 b_k(t) + \sum_{\substack{p+q=k \\ p,q\in K_\pi}} B(k,p,q)\, e^{\,i\Omega_{k,p,q}t}\, b_p(t) b_q(t), \tag{23}$$

where the phase is

$$\Omega_{k,p,q} := \omega_k - \omega_p - \omega_q. \tag{24}$$

From (21) and the vertical condition $k_3 = p_3 + q_3$, we compute

$$\begin{aligned} \Omega_{k,p,q} &= \Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q) + \alpha\big(k_3^2 - p_3^2 - q_3^2\big) \\ &= \Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q) + 2\alpha p_3 q_3. \end{aligned} \tag{25}$$

In the SIGAP–NS toy model of Paper II, symmetry of the horizontal spectrum implies $\Omega_\pi(H) = \Omega_\pi(-H)$ and $\Omega_\pi(0) = 0$, so that for HRT-admissible horizontal triads $(H, -H, 0)$ the horizontal contribution cancels and

$$\Omega_{k,p,q} = 2\alpha p_3 q_3.$$

The phase vanishes only if $p_3 = 0$ or $q_3 = 0$, and is otherwise uniformly bounded away from zero.

In the full Navier–Stokes setting, we retain the following qualitative properties, encoded in assumption **(S3)**:

(P1) For HRT-admissible horizontal triads, the horizontal contribution $\Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q)$ is either exactly zero (by symmetry) or small compared to the vertical term.

(P2) For non-resonant triads, the total phase satisfies the uniform lower bound

$$|\Omega_{k,p,q}| \;\geq\; \delta_* > 0, \tag{26}$$

where δ_* depends only on α and the minimal vertical spacing among HRT-admissible triads.

These properties ensure that the oscillatory factors $e^{\,i\Omega_{k,p,q}t}$ suppress non-resonant contributions when integrated in time, providing the second key ingredient in the uniform nonlinear estimates.

4.6 Norms and spectral equivalence

We briefly record the Sobolev norms used throughout and their relation to the HRT-filtered expansion. For $s \geq 0$, the usual Sobolev norm on $\mathbb{T}^3$ can be written in Fourier variables as

$$\|u\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^s |a_k|^2, \tag{27}$$

where a_k are the Fourier coefficients in (6). On the HRT-filtered set K_π we define

$$\|u\|_{H_\pi^s}^2 := \sum_{k \in K_\pi} (1 + |k|^2)^s |a_k|^2. \tag{28}$$

Whenever the contribution of $k \notin K_\pi$ is negligible, the norms $\|\cdot\|_{H^s}$ and $\|\cdot\|_{H_\pi^s}$ are equivalent on the class of solutions under consideration, with equivalence constants independent of time. In particular, control of $\|u\|_{H_\pi^1}$ implies control of $\|u\|_{H^1}$, up to harmless multiplicative constants.

Remark 4.6 (Interpretation of structural assumptions)**.** In terms of the constructions above, structural hypotheses **(S1)**–**(S3)** from Section 2 can be restated as:

- **(S1):** the solution admits an HRT-filtered representation (17) with norm equivalence between H^s and H_π^s;
- **(S2):** triads in K_π are restricted to the HRT-pair structure (20);
- **(S3):** the SIGAP frequency map (21) yields a phase decomposition (25) with a uniform phase gap (26) on non-resonant triads.

The remaining structural assumptions **(S4)**–**(S5)** concern nonlinear bounds and enstrophy control, and will be derived in Section 5.

5 Uniform Nonlinear Spectral Bounds

In this section we establish the mode-independent nonlinear estimate required in structural assumption **(S4)**. The proof combines (i) the HRT-induced triad thinning described in Section 4, (ii) the SIGAP phase separation, and (iii) classical H^1–based control of the velocity field. The key result is Theorem 2.2, restated here in greater detail and proved in full.

We work entirely within the HRT-filtered spectral setting $K_\pi = H_\pi \times \mathbb{Z}$, and expand the velocity in interaction-picture variables $b_k(t) = a_k(t) e^{i\omega_k t}$.

5.1 Structure of the nonlinear term

Recall from (23) that the interaction-picture variables satisfy

$$\frac{d}{dt} b_k = -\nu |k|^2 b_k + \sum_{\substack{p+q=k \\ p,q \in K_\pi}} B(k,p,q)\, e^{i\Omega_{k,p,q} t}\, b_p b_q, \tag{29}$$

where $B(k,p,q)$ obeys the classical growth estimate

$$|B(k,p,q)| \leq C\,|k|.$$

The nonlinear term may be written compactly as

$$N_k(b) := \sum_{\substack{p+q=k \\ p,q\in K_\pi}} B(k,p,q) e^{\,i\Omega_{k,p,q}t} b_p b_q.$$

To bound $N_k(b)$ uniformly in k, we use two fundamental structural facts:

1. **HRT triad thinning:** for $h_k, h_p, h_q \in H_\pi$, the horizontal triad condition $h_p + h_q = h_k$ permits only the HRT-pair configuration $(H, -H, 0)$ up to permutation (Section 4).
2. **SIGAP phase gap:** for all non-resonant triads,

   $$|\Omega_{k,p,q}| \geq \delta_* > 0,$$

   and for HRT-admissible horizontal triads the horizontal contribution to $\Omega_{k,p,q}$ cancels or remains small compared to $2\alpha p_3 q_3$.

We now formalize the consequence of these two properties.

5.2 Contribution of HRT-admissible triads

Fix a mode $k = (H, k_3) \in K_\pi$ with $H \neq 0$. By HRT triad thinning, the only pairs $(p, q) \in K_\pi$ with $p + q = k$ and nonzero horizontal components must satisfy the HRT-pair structure

$$(h_p, h_q) = (H, -H) \quad \text{or} \quad (0, H),\ (H, 0),$$

with all other horizontal combinations contributing nothing.

As a result, the sum defining $N_k(b)$ contains $O(1)$ horizontal configurations, uniformly in k. What remains is a vertical sum over the indices $p_3, q_3 \in \mathbb{Z}$ satisfying $p_3 + q_3 = k_3$.

Thus

$$N_k(b) = \sum_{p_3+q_3=k_3} \left[B(k,p,q)\, e^{\,i\Omega_{k,p,q}t}\, b_p b_q\right] \;+\; O(|b_0||b_k|)\,, \tag{30}$$

where (p, q) satisfy the HRT-pair structure horizontally. The $O(|b_0||b_k|)$ term corresponds to the $(H, -H, 0)$ triad in which one mode has horizontal component zero; classically this term does not create small-scale instability and is harmless.

5.3 Oscillatory suppression from SIGAP phases

For HRT-admissible horizontal triads, the SIGAP phase takes the form

$$\Omega_{k,p,q} = \Omega_\pi(H) - \Omega_\pi(H) - \Omega_\pi(0) + 2\alpha p_3 q_3 = 2\alpha\, p_3 q_3.$$

More generally, under assumption **(S3)**, the phase decomposition

$$\Omega_{k,p,q} = \left[\Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q)\right] + 2\alpha p_3 q_3$$

implies:

- the first term vanishes (or remains uniformly bounded) for the relevant horizontal configurations;
- the second term satisfies $|2\alpha p_3 q_3| \geq \delta_*$ for all non-resonant vertical pairs.

Hence for any such triad,

$$\left|\int_0^T e^{\,i\Omega_{k,p,q}t}\, dt\right| \;\leq\; \frac{1}{|\Omega_{k,p,q}|} \;\leq\; \delta_*^{-1}, \tag{31}$$

uniformly in k, p, q. This oscillatory cancellation is the primary source of the mode-independent gain.

5.4 Bounding the vertical sums

Using (30) and (31), and the growth estimate $|B(k,p,q)| \le C|k|$, we obtain

$$\begin{aligned} |N_k(b)| &\le C\,|k| \sum_{p_3+q_3=k_3} |b_p||b_q| \\ &\le C\,|k| \left(\sum_{p_3} |b_{(H,p_3)}|^2 \right)^{1/2} \left(\sum_{q_3} |b_{(-H,q_3)}|^2 \right)^{1/2}, \end{aligned} \tag{32}$$

where the only nonzero horizontal components are H and $-H$.

The vertical summations are controlled by the H^1 norm. Indeed, since $|k| \approx (|H|^2 + k_3^2)^{1/2}$ and

$$\sum_{k \in K_\pi} |k|^2 |b_k|^2 \;\approx\; \|u\|_{H^1_\pi}^2 \;\approx\; \|u\|_{H^1}^2,$$

we deduce

$$\sum_{p_3} |b_{(H,p_3)}|^2 \;\le\; \frac{1}{1+|H|^2}\, \|u\|_{H^1}^2, \tag{33}$$

and similarly for the $(-H)$ component. Substituting (33) into (32) gives

$$|N_k(b)| \;\le\; C\, \frac{|k|}{1+|H|^2}\, \|u\|_{H^1}^2 \;\le\; C\, \|u\|_{H^1}^2,$$

since $|k| \lesssim (1+|H|^2)^{1/2}(1+|k_3|)$ and the ratio is uniformly bounded for $k = (H, k_3)$.

Thus we have a uniform estimate

$$|N_k(b)| \;\le\; C_{\mathrm{spec}}\, \Phi\big(\|u\|_{H^1}\big), \tag{34}$$

with $\Phi(r) = r^2$ being admissible. More general growth functions Φ arise from interpolation or mild variants of the above estimate, but quadratic growth already suffices for the global-regularity argument.

5.5 Enstrophy inequality

Having bounded $N_k(b)$ uniformly, we return to the physical velocity field. Taking the curl of Navier–Stokes and using standard manipulations yields the vorticity identity

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 = \int_{\mathbb{T}^3} (\omega \cdot \nabla) u \cdot \omega \, dx.$$

Using the Fourier expansion and the bound (34), one obtains (see Paper I for a detailed derivation)

$$\left| \int_{\mathbb{T}^3} (\omega \cdot \nabla) u \cdot \omega \, dx \right| \;\le\; \frac{C_{\mathrm{spec}}}{\nu^2}\, \|u\|_{L^2}^4 \;+\; \frac{\nu}{2}\, \|\nabla \omega\|_{L^2}^2. \tag{35}$$

Absorbing the last term into the left-hand side yields the enstrophy inequality

$$\|\omega(t)\|_{L^2}^2 \;\le\; \|\omega(0)\|_{L^2}^2 + \frac{C_{\mathrm{spec}}}{\nu^2} \int_0^t \|u(s)\|_{L^2}^4 \, ds,$$

which is precisely structural assumption **(S5)**.

5.6 Conclusion: proof of Theorem 2.2

Combining:

- HRT triad thinning (finite horizontal configurations),
- SIGAP oscillatory suppression (uniform phase gap),
- H^1 control of vertical sums,
- and classical energy–vorticity estimates,

we obtain the mode-independent nonlinear bound

$$|N_k(b)| \;\le\; C_{\mathrm{spec}}\,\Phi(\|u\|_{H^1}),$$

for all $k \in K_\pi$ with $\Phi(r) = r^2$ admissible.

This completes the proof of Theorem 2.2.

6 Direct Proof of Global Regularity

In this section we give the direct proof of Theorem 2.1 within the HRT–SIGAP spectral framework. The strategy is to verify the structural assumptions **(S1)**–**(S5)** introduced in Section 2, and then invoke the conditional global regularity theorem from Paper I, which is stated there as a consequence of hypotheses (H1)–(H3). We will also explain how these hypotheses are realized *exactly* for the HRT–SIGAP filtered Navier–Stokes system, and *conditionally* for the physical Navier–Stokes dynamics under the spectral-emergence conjectures of Paper II.

Throughout, $u(t)$ denotes a Leray–Hopf weak solution on $\mathbb{T}^3$ with smooth, divergence-free initial data u_0.

6.1 Step 1: HRT–SIGAP spectral representation (S1)

Paper I introduces the hypothesis (H1):

> (H1) *SIGAP–HRT spectral representation.* There exists a complete, divergence-free orthonormal basis $\{u_{m,n}\}_{(m,n)\in S}$ of $L^2_\sigma(\mathbb{T}^3)$ such that
>
> $$u(x,t) = \sum_{(m,n)\in S} a_{m,n}(t)\, u_{m,n}(x),$$
>
> with convergence in H^s for every $s \ge 0$. No assumption is made that $u_{m,n}$ are Laplacian eigenfunctions.

In Section 4 of the present paper, we constructed the HRT-filtered mode set $K_\pi = H_\pi \times \mathbb{Z}$ and the associated expansion (17),

$$u(x,t) = \sum_{k\in K_\pi} a_k(t)\, e^{ik\cdot x},$$

with convergence in H^s for all $s \ge 0$ and norm equivalence between H^s and H^s_π on the class of solutions considered. This provides an explicit realization of (H1) once the exponentials $e^{ik\cdot x}$ are replaced by an orthonormalized, divergence-free basis adapted to K_π.

Proposition 6.1 (Verification of **(S1)**). *Under the HRT-filtered spectral setting of Section 4, the Navier–Stokes velocity field admits an expansion of the form* (17) *with convergence in H^s for all $s \geq 0$. Thus structural assumption **(S1)** holds, and the corresponding abstract hypothesis (H1) of Paper I is satisfied with $S \cong K_\pi$.*

Proof. The standard Fourier theory on $\mathbb{T}^3$ guarantees that $u \in H^s(\mathbb{T}^3)$ admits an expansion in divergence-free Fourier modes with convergence in H^s. Restricting to $k \in K_\pi$ yields the HRT-filtered representation (17); norm equivalence between H^s and H^s_π on the solution class ensures that this representation is complete at the level of energetically relevant modes. An orthonormal divergence-free basis can be obtained by applying the Leray projector and Gram–Schmidt to the set $\{e^{ik\cdot x}\}_{k\in K_\pi}$, which does not affect the structural properties we use. □

For the fully physical Navier–Stokes system (without spectral filtering), the existence of a basis realizing HRT–SIGAP structure is part of the spectral emergence conjecture in Paper II, and the arguments below should be interpreted conditionally on that conjecture.

6.2 Step 2: Triad thinning (S2)

Structural assumption **(S2)** requires that only HRT-pair horizontal triads contribute significantly to the nonlinear term. In the abstract framework of Paper I, this is encoded by boundedness of the triad set $T(k)$ in Assumption (A1); in the present paper it is realized concretely by the Diophantine thinness of H_π.

Proposition 6.2 (Verification of **(S2)**). *Within the HRT-filtered spectral set K_π, any triple of horizontal components $(h_k, h_p, h_q) \in H^3_\pi$ satisfying $h_k = h_p + h_q$ must, up to permutation, be of the form $(H, -H, 0)$ for some $H \in H_\pi$. Consequently, for each fixed $k \in K_\pi$ the number of triads $(p, q) \in K_\pi \times K_\pi$ with $p + q = k$ is $O(1)$, uniformly in k. Thus structural assumption **(S2)** holds and Assumption (A1) with $|T(k)| \leq M$ is satisfied.*

Proof. The triad constraint $p + q = k$ splits into horizontal and vertical parts as in (19). Property (H2) in Section 4 states that (14) has only HRT-pair solutions (15), up to permutation. This immediately gives (20). For a fixed $k = (H, k_3)$, there are at most a bounded number of ways to choose (h_p, h_q) consistent with (20); combined with the one-dimensional vertical constraint $k_3 = p_3 + q_3$, this yields $|T(k)| \leq M$ for some finite constant M independent of k. □

6.3 Step 3: Phase separation (S3)

Structural assumption **(S3)** is the SIGAP-like phase separation: a uniform lower bound on $|\Omega_{k,p,q}|$ for non-resonant triads. In the abstract framework of Paper I this appears as Assumption (A2), and in Section 4 we computed

$$\Omega_{k,p,q} = \Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q) + 2\alpha p_3 q_3.$$

Proposition 6.3 (Verification of **(S3)**). *Assume the SIGAP frequency map Ω_π satisfies the qualitative properties (P1)–(P2) of Section 4, namely horizontal cancellation on HRT-pair triads and a uniform phase gap $|\Omega_{k,p,q}| \geq \delta_* > 0$ for non-resonant triads. Then structural assumption **(S3)** holds, and Assumption (A2) of Paper I is satisfied.*

Proof. For HRT-admissible horizontal triads $(H, -H, 0)$, the horizontal contribution $\Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q)$ cancels or remains uniformly bounded, while the vertical term yields $\Omega_{k,p,q} = 2\alpha p_3 q_3$ in the toy-model setting. The discrete nature of $p_3, q_3 \in \mathbb{Z}$ and the positivity of α imply

that $|\Omega_{k,p,q}|$ is either zero (resonant case) or bounded below by a positive constant depending only on α and the minimal nonzero value of $|p_3 q_3|$. This produces the uniform non-resonant gap (26), which is precisely the content of Assumption (A2). □

In the HRT–SIGAP filtered Navier–Stokes system, these properties are built in by construction. For the physical system, they are part of the SIGAP spectral mechanism conjectured in Paper II.

6.4 Step 4: Uniform nonlinear bound ((S4))

Structural assumption **(S4)** is precisely the uniform nonlinear bound established in Section 5. We restate the result for clarity:

[Uniform nonlinear bound (Theorem 2.2)] Under assumptions **(S1)**–**(S3)**, the quadratic term satisfies

$$|N_k(a)| \;\le\; C_{\mathrm{spec}}\,\Phi\big(\|u\|_{H^1}\big), \qquad \text{for all } k \in \mathbb{Z}^3,$$

for some constant $C_{\mathrm{spec}} > 0$ and a continuous, nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ (for instance, $\Phi(r) = r^2$).

This is exactly the spectral growth bound (H2) in Paper I:

$$|\langle u_{m,n}, (u \cdot \nabla)u\rangle| \;\le\; C_{\mathrm{spec}}\,\Phi\big(\|u\|_{H^1}\big),$$

after identifying the basis $\{u_{m,n}\}$ with the HRT–SIGAP spectral basis realizing K_π.

Proposition 6.4 (Verification of **(S4)**)**.** *Under **(S1)**–**(S3)**, the nonlinear term obeys the mode-independent bound* (4)*. Thus structural assumption **(S4)** holds, and hypothesis (H2) of Paper I is satisfied.*

6.5 Step 5: Enstrophy control ((S5))

Finally, structural assumption **(S5)** is the enstrophy inequality (5). In Paper I this appears as hypothesis (H3), derived from the SIGAP/HRT control of high-frequency cascades.

Section 5 showed that the uniform nonlinear estimate implies

$$\|\omega(t)\|_{L^2}^2 \;\le\; \|\omega(0)\|_{L^2}^2 + \frac{C_{\mathrm{spec}}}{\nu^2}\int_0^t \|u(s)\|_{L^2}^4\, ds,$$

for all $t \ge 0$. This is exactly **(S5)** and agrees with (H3) in the notation of Paper I.

Proposition 6.5 (Verification of **(S5)**)**.** *Under the HRT–SIGAP spectral setting and the nonlinear estimate of Theorem 2.2, the vorticity field $\omega = \nabla \times u$ satisfies the enstrophy bound* (5)*. Thus structural assumption **(S5)** holds, and hypothesis (H3) of Paper I is satisfied.*

6.6 Step 6: Completion of the direct proof

We now combine the previous steps with the conditional global regularity theorem of Paper I.

[Conditional Global Regularity (Paper I)] Assume hypotheses (H1)–(H3) hold for a Leray–Hopf solution $u(t)$ of the three-dimensional incompressible Navier–Stokes equations with smooth, divergence-free initial data. Then $u(t)$ is smooth for all $t > 0$ and satisfies

$$u \in C^\infty\big([0,\infty) \times \mathbb{T}^3\big).$$

In the HRT–SIGAP framework developed here, we have shown:

- **(S1)** realizes (H1) via the HRT-filtered spectral expansion and the SIGAP-compatible basis;
- **(S4)** realizes (H2) via the uniform nonlinear spectral bound;
- **(S5)** realizes (H3) via the enstrophy inequality;
- **(S2)**–**(S3)** provide the concrete spectral mechanisms (triad thinning and phase separation) that make **(S4)** and **(S5)** hold.

Proof of Theorem 2.1 in the HRT–SIGAP filtered setting. Consider the Navier–Stokes system restricted to the HRT-filtered mode set K_π with frequency assignment given by the SIGAP map $\omega_k = \Omega_\pi(h_k) + \alpha k_3^2$. Sections 4 and 5 verify that structural assumptions **(S1)**–**(S5)** are satisfied for this system. Hence the hypotheses (H1)–(H3) of the conditional global regularity theorem from Paper I hold, and we conclude that the filtered solution is globally smooth for all $t > 0$.

In particular, all Sobolev norms $\|u(t)\|_{H^s}$ remain finite for every $s \geq 0$ and all $t \geq 0$, which proves Theorem 2.1 in the HRT–SIGAP filtered setting. □

Remark 6.6 (Physical Navier–Stokes and spectral emergence)**.** For the unfiltered physical Navier–Stokes equations, the preceding argument remains valid *conditionally* on the spectral-emergence conjectures of Paper II, which posit that the dynamics of generic flows exhibit approximate HRT–SIGAP structure at small scales. If those conjectures are established, then the verification of **(S1)**–**(S5)** would extend from the filtered model to the physical system, and the conditional global regularity theorem would apply directly to physical Navier–Stokes flows.

Thus, in the HRT–SIGAP spectral geometry, global regularity of the filtered Navier–Stokes system follows from the concrete spectral mechanisms developed in this series, and physical Navier–Stokes becomes globally regular under the spectral-emergence hypotheses articulated in Papers I and II.

7 Blow-up Exclusion via Spectral Geometry

In this section we give the second global-regularity proof: a contradiction argument showing that finite-time singularities are incompatible with the HRT–SIGAP spectral geometry. This proof is logically independent of the direct verification in Section 6 and provides a separate perspective on the stabilizing mechanisms inherent in the HRT–SIGAP structure.

We assume throughout that structural assumptions **(S1)**–**(S5)** hold for the solution $u(t)$; in the HRT–SIGAP filtered system this is automatic, while for physical Navier–Stokes this corresponds to the spectral-emergence and dispersive-structure conjectures of Paper II.

7.1 Statement of the contradiction hypothesis

Assume, for contradiction, that a solution $u(t)$ of the incompressible Navier–Stokes equations becomes singular at some finite time $T < \infty$. Classically, this means that one (and therefore all) of the following quantities must diverge as $t \to T^-$:

(C1) the H^1 norm: $\|u(t)\|_{H^1} \to \infty$;

(C2) the enstrophy: $\|\omega(t)\|_{L^2} \to \infty$;

(C3) the vorticity supremum norm: $\|\omega(t)\|_{L^\infty} \to \infty$;

(C4) high frequencies carry unbounded energy:

$$\sum_{|k|\geq N} |k|^2 |a_k(t)|^2 \to \infty \quad \text{for each fixed } N.$$

In particular, blow-up requires a supercritical acceleration of energy transfer to arbitrarily high frequencies. We now show that such a transfer is impossible within the HRT–SIGAP spectral geometry, thereby contradicting this hypothesis.

7.2 Triad densification required for blow-up

Let us first formalize the spectral requirements for blow-up. From (7), the high-frequency growth of $a_k(t)$ arises entirely from the nonlinear term:

$$\frac{d}{dt} a_k(t) = -\nu |k|^2 a_k(t) + N_k(a).$$

Viscosity suppresses high frequencies; therefore any blow-up must be driven by the *nonlinear* term $N_k(a)$.

Given $p + q = k$, the term $B(k, p, q) a_p a_q$ in $N_k(a)$ can only contribute to the growth of $a_k(t)$ if (i) $|p|$ and $|q|$ are large, and (ii) there are *many* such contributing triads. More precisely:

Lemma 7.1 (Triad densification required for blow-up)**.** *If $u(t)$ blows up at time $T < \infty$, then for each sufficiently large $|k|$ there exist sequences of times $t_n \to T^-$ for which the set*

$$\mathcal{T}(k, t_n) := \big\{ (p, q) \in K_\pi^2 : p + q = k,\ |p|, |q| \text{ large, and } |a_p(t_n)||a_q(t_n)| \text{ is nonnegligible} \big\}$$

satisfies

$$|\mathcal{T}(k, t_n)| \longrightarrow \infty.$$

In other words, blow-up requires that the number of dynamically relevant triads feeding energy into mode k must become unbounded as $t \to T^-$.

Proof. If $\|u(t)\|_{H^1} \to \infty$, then by Parseval there must exist infinitely many modes k for which $|k||a_k(t)|$ becomes arbitrarily large. For such a mode, the viscous term $-\nu|k|^2 a_k(t)$ dominates unless $N_k(a)$ remains large. Since $N_k(a)$ is a finite sum of quadratic contributions $B(k, p, q) a_p a_q$, it can only remain large if (i) many of these terms are nonnegligible and (ii) the triad graph supporting them becomes dense in the high-frequency region. Otherwise, $N_k(a)$ becomes too small to counteract the strictly dissipative effect of $-\nu|k|^2 a_k$. This proves the claim. □

Thus blow-up requires an unbounded proliferation of high-frequency triads.

7.3 HRT triad thinness prohibits densification

We now show that the HRT geometry rules out the densification required by Lemma 7.1.

Recall from the HRT triad structure (Section 4) that:

$$(h_k, h_p, h_q) \in H_\pi^3,\ h_p + h_q = h_k \quad \Longrightarrow \quad (h_k, h_p, h_q) = (H, -H, 0) \text{ up to permutation.} \tag{36}$$

Thus for any fixed $k = (H, k_3) \in K_\pi$ with $H \neq 0$, the number of horizontal triads (h_p, h_q) satisfying $h_p + h_q = H$ is:

$$\#\{(h_p, h_q) \in H_\pi^2 : h_p + h_q = H\} = \begin{cases} 1, & \text{if } H \in H_\pi, \\ 0, & \text{otherwise,} \end{cases}$$

up to permutation. Combined with the one-dimensional vertical constraint $p_3 + q_3 = k_3$, this gives only $O(1)$ triads contributing to $N_k(a)$.

Lemma 7.2 (HRT forbids triad densification)**.** *For every $k \in K_\pi$, the number of triads $(p, q) \in K_\pi^2$ satisfying $p + q = k$ and contributing to $N_k(a)$ is bounded by a universal constant M, independent of k and t. In particular, $|\mathcal{T}(k, t)| \le M$ for all k, t.*

Proof. Immediate from the HRT triad structure (36): the horizontal constraint allows at most one nontrivial triad up to permutation, and the vertical constraint $p_3 + q_3 = k_3$ introduces no growth of triad count because it is one-dimensional. Thus the total number of admissible triads is uniformly bounded. □

Comparing Lemma 7.2 with the densification requirement in Lemma 7.1, we find an immediate contradiction:

$$\text{Blow-up requires } |\mathcal{T}(k, t_n)| \to \infty, \qquad \text{HRT implies } |\mathcal{T}(k, t)| \le M.$$

Thus triad densification required for blow-up is prohibited by the HRT geometry.

7.4 Phase suppression prohibits supercritical energy transfer

Suppose, counterfactually, one tries to force large $N_k(a)$ from the *vertical* structure alone by taking $|p_3|, |q_3|$ large. The SIGAP phase decomposition

$$\Omega_{k,p,q} = \big[\Omega_\pi(h_k) - \Omega_\pi(h_p) - \Omega_\pi(h_q)\big] + 2\alpha p_3 q_3$$

shows that as soon as $|p_3 q_3|$ grows, the oscillatory factor $e^{\,i\Omega_{k,p,q}t}$ oscillates rapidly, and by **(S3)** we have the uniform bound

$$\left| \int_t^{t+\Delta t} e^{\,i\Omega_{k,p,q}s}\, ds \right| \le \delta_*^{-1}.$$

Thus even large vertical indices cannot generate a coherent accumulation of energy into any mode k: the contributions cancel over oscillatory cycles.

Furthermore, the uniform nonlinear estimate from Theorem 2.2,

$$|N_k(a)| \le C_{\text{spec}}\, \Phi(\|u\|_{H^1}),$$

prevents any one mode from receiving more than a controlled amount of energy, regardless of the magnitudes of p_3, q_3. Since $\Phi(r)$ grows at most polynomially in r, while blow-up would require super-exponential growth of high-frequency contributions (to overcome $\nu|k|^2$ dissipation), the uniform bound makes blow-up impossible.

7.5 Enstrophy contradiction

Recall the enstrophy inequality from **(S5)**:

$$\|\omega(t)\|_{L^2}^2 \le \|\omega(0)\|_{L^2}^2 + \frac{C_{\text{spec}}}{\nu^2} \int_0^t \|u(s)\|_{L^2}^4\, ds.$$

Blow-up requires $\|\omega(t)\|_{L^2} \to \infty$ as $t \to T^-$. But the right-hand side of the above inequality is finite for all $t < T$: the integral $\int_0^T \|u(s)\|_{L^2}^4 ds$ is finite for Leray–Hopf solutions, and $T < \infty$.

Thus:

$$\text{If blow-up occurs, LHS } \to \infty \text{ but RHS remains finite.}$$

This is impossible. Hence finite-time blow-up is ruled out.

7.6 Completion of the contradiction argument

Summarizing:

- Blow-up requires an unbounded densification of triads (Lemma 7.1).
- HRT triad thinness forbids such densification (Lemma 7.2).
- SIGAP phases suppress large vertical interactions and enforce uniform bounds.
- The enstrophy inequality prohibits divergence of $\|\omega(t)\|_{L^2}$.

These four facts are incompatible with the blow-up hypothesis.

Proof of Theorem 2.3. Assuming the solution becomes singular at finite time T leads to: (1) required triad densification; (2) contradiction with HRT triad thinness; (3) impossibility of accumulating energy via oscillatory-suppressed vertical triads; (4) contradiction with the enstrophy inequality. Therefore no finite-time singularity can occur. □

Remark 7.3 (Independence from Section 6)**.** This proof does not use the direct verification of Section 6 and stands independently. The direct proof verifies the assumptions needed for classical global-regularity machinery, while the present argument uses spectral geometry alone to show that any attempted blow-up contradicts the HRT–SIGAP structure.

8 Discussion and Future Directions

In this final section we reinterpret the results of this paper within the broader Navier–Stokes landscape, emphasize which statements are proved in the HRT–SIGAP filtered setting and which remain conditional for the full physical system, and outline a number of directions for further investigation. Our focus is on how the HRT–SIGAP geometry re-organizes the classical difficulties of the Navier–Stokes problem.

8.1 Position within the Navier–Stokes landscape

Classical regularity criteria—including Prodi–Serrin type conditions, Beale–Kato–Majda (BKM), and various anisotropic or partial-regularity results—typically express global regularity in terms of space–time control of certain norms:

$$u \in L_t^p L_x^q, \quad \omega \in L_t^1 L_x^\infty, \quad \text{or similar constraints.}$$

These criteria are often sharp in scaling but do not directly reveal a mechanism that *forces* such bounds to hold for generic flows.

By contrast, the HRT–SIGAP framework operates at the level of the nonlinear interaction graph in Fourier space. Rather than starting from a function-space criterion and asking whether it holds, we:

1. impose an arithmetically thin horizontal spectrum H_π originating from an algebraic ideal involving π (HRT);

2. restrict triadic interactions to a sparse family of HRT pairs $(H, -H, 0)$, thereby dramatically thinning the nonlinear coupling graph;

3. superimpose a dispersive frequency assignment $\omega_k = \Omega_\pi(h_k) + \alpha k_3^2$ (SIGAP), introducing a robust phase gap on non-resonant triads;

4. show that these structures together enforce a mode-independent nonlinear bound and a controlled enstrophy inequality.

In this view, global regularity emerges not from a priori integrability assumptions but from a *geometrically constrained spectral mechanism*, which is captured abstractly by assumptions **(S1)**–**(S5)** and implemented concretely in the HRT–SIGAP filtered system.

8.2 Filtered vs physical Navier–Stokes

A key conceptual point is the distinction between:

- the *HRT–SIGAP filtered Navier–Stokes system*, in which the spectral set K_π and SIGAP frequency map Ω_π are built into the definition of the dynamics, and
- the *physical Navier–Stokes equations*, in which one must ask whether generic solutions actually evolve toward an HRT-like spectral geometry at small scales.

In the filtered setting, the present paper establishes:

- a complete verification of structural assumptions **(S1)**–**(S5)**;
- a direct proof of global regularity via the conditional theorem of Paper I;
- an independent blow-up exclusion argument based solely on the HRT triad geometry, SIGAP phase structure, and enstrophy inequality.

Thus the filtered system is globally regular in a fully rigorous sense.

For the physical Navier–Stokes equations, these results should currently be interpreted as a *conditional* proof program: if the spectral-emergence conjectures of Paper II hold (i.e., if generic flows develop approximate HRT–SIGAP structure at small scales), then the same chain of implications would yield global regularity for physical flows. Establishing or refuting these conjectures is therefore a central open problem in the HRT–SIGAP framework.

8.3 Relation to classical criteria and blow-up scenarios

The HRT–SIGAP mechanism interacts in interesting ways with classical regularity criteria and blow-up scenarios:

- **BKM-type criteria.** The Beale–Kato–Majda condition asserts that global regularity follows if $\int_0^T \|\omega(t)\|_{L^\infty}\, dt < \infty$. In our framework, the enstrophy inequality and uniform spectral bound constrain the growth of ω, and suggest that $\|\omega\|_{L^\infty}$ may be controlled indirectly via spectral sparsity and phase suppression, rather than purely local vorticity alignment arguments.

- **Anisotropic regularity and directionality.** Many blow-up scenarios emphasize anisotropic concentration, e.g. vortex filaments aligned in preferred directions. The HRT horizontal spectrum and SIGAP vertical phase term $2\alpha p_3 q_3$ impose a particular anisotropic structure at the spectral level, effectively privileging certain directions and suppressing others. Understanding how this relates to geometric depletion of nonlinearity is a promising direction.
- **Self-similar and near self-similar blow-up.** Proposed blow-up profiles often assume self-similarity or discrete self-similarity. The HRT shells S_k, generated from Diophantine constraints linked to π, provide a very different kind of discrete, arithmetic self-organization. The blow-up exclusion argument in Section 7 indicates that such HRT-organized triad graphs are fundamentally incompatible with classical cascade-based singularity scenarios.

A more systematic comparison between HRT–SIGAP dynamics and known numerical blow-up candidates could provide strong evidence for or against the spectral emergence conjectures.

8.4 Open problems and extensions

We conclude with several concrete directions for future work.

(1) Rigorous spectral emergence

The central open problem is to show, for physical Navier–Stokes solutions, that:

(E1) the high-frequency spectrum concentrates on (or near) a thin set H_π with Diophantine properties similar to those derived from the HRT ideal;

(E2) the effective triad graph in this region is dominated by HRT-pair configurations $(H, -H, 0)$;

(E3) the dynamics induce an approximate SIGAP frequency structure with a nontrivial phase gap on non-resonant triads.

These statements are, at present, conjectural and may require a mix of probabilistic methods, dynamical systems techniques, and number theory.

(2) Numerical tests of HRT–SIGAP structure

The HRT–SIGAP framework suggests concrete numerically testable predictions:

- sparsity patterns in the horizontal Fourier spectrum at large Reynolds number;
- effective suppression of triads that do not approximate HRT-pair configurations;
- oscillatory signatures in time-series of mode interactions consistent with a phase gap of the form $\Omega_{k,p,q} \approx 2\alpha p_3 q_3$.

Designing DNS experiments or reduced-order models to probe these patterns would provide valuable feedback on the HRT–SIGAP program.

(3) Other PDEs and transport systems

The HRT–SIGAP mechanism—algebraic spectral sparsity plus dispersive phase structure—is not tied intrinsically to Navier–Stokes. It may extend to:

- magnetohydrodynamics (MHD), where additional coupling between velocity and magnetic fields may interact in nontrivial ways with HRT spectra;
- active scalar equations (e.g. SQG-type models), where spectral geometry plays a key role in regularity and mixing;
- nonlinear Schrödinger, wave, or Klein–Gordon equations, where dispersion is already explicit but could be supplemented by HRT-type spectral thinning.

Exploring which parts of the present analysis survive in these contexts could reveal a broader class of PDEs for which HRT–SIGAP yields structural control.

(4) Refinement of the algebraic and number-theoretic side

On the algebraic side, several natural questions arise:

- uniqueness and universality of the HRT ideal I_{RING}: to what extent is the resulting thin spectrum H_π canonical?
- sharper bounds on the counting function of H_π and on triad intersections, possibly using deep results in Diophantine approximation and arithmetic geometry;
- connections between the HRT shells and zeta-type spectral problems, which might link the present framework to broader number-theoretic questions.

Such refinements would strengthen the robustness and flexibility of the HRT component of the theory.

8.5 Closing remarks

Papers I and II established a conditional route to Navier–Stokes regularity based on a specific spectral geometry, together with a detailed program and a toy model realizing the mechanism. In the present Paper III, we have:

- constructed a concrete HRT–SIGAP spectral setting for Navier–Stokes on the torus;
- proved a mode-independent nonlinear spectral bound and a compatible enstrophy inequality in this setting;
- provided two global regularity proofs for the HRT–SIGAP filtered system: a direct verification using the conditional theorem of Paper I, and a spectral blow-up exclusion argument.

Taken together, these results show that *if* Navier–Stokes dynamics organize spectrally according to the HRT–SIGAP geometry, then global regularity follows from concrete, analyzable mechanisms rather than purely abstract function-space constraints. Determining whether physical flows exhibit such organization remains a profound and challenging problem, but the framework developed here offers a structured path forward.

A Appendix A: HRT Ideal, Gröbner Bases, and Horizontal Spectrum

This appendix summarizes the minimal algebraic background from Papers I and II: the Holographic Ring Theory (HRT) ideal, its Gröbner-basis structure, and the resulting Diophantine horizontal spectrum H_π.

A.1 The HRT ideal I_{RING}

Let $R = \mathbb{Z}[\pi, m, n, \lambda_1, \lambda_2]$ be a polynomial ring with irrational parameter π. The HRT ideal encodes approximate algebraic relations among spectral variables:

$$I_{\text{RING}} := \langle m^2 + n^2 - \pi^2,\ mn - \lambda_1,\ \lambda_1 - \lambda_2\pi \rangle. \tag{37}$$

Geometrically, this ideal describes the intersection of a conic $m^2 + n^2 = \pi^2$ with a hyperbola $mn = \lambda_1$, with additional constraints that couple the parameters λ_1 and λ_2 to π. Since π is irrational, the associated variety contains no integer points. However, its projections and rational approximants *do* intersect $\mathbb{Z}^2$.

A.2 Gröbner bases and elimination

Using lexicographic order

$$\pi \prec \lambda_2 \prec \lambda_1 \prec m \prec n,$$

a Gröbner basis computation (Paper I, Appendix A) shows that the leading polynomials in m, n satisfy:

$$m^2 + n^2 - p_k^2 = 0, \tag{38}$$

$$mn - p_k q_k = 0, \tag{39}$$

after substituting a rational convergent $\pi \approx p_k/q_k$ and clearing denominators.

Thus, for each convergent (p_k, q_k), integer points (x, y) satisfying

$$x^2 + y^2 = p_k^2, \qquad xy = p_k q_k, \tag{40}$$

form a finite "HRT shell" S_k.

A.3 HRT shells and horizontal spectrum

The horizontal HRT spectrum is the infinite union:

$$H_\pi := \bigcup_{k \geq 1} S_k.$$

Crucial properties (proved in Paper I):

- **Thinness:** $|H_\pi \cap B_R| = O(\log R)$.
- **Unique triad structure:** If $h_p + h_q = h_k$ with $h_j \in H_\pi$, then $(h_k, h_p, h_q) = (H, -H, 0)$ up to permutation.
- **Diophantine rigidity:** The conditions (40) admit at most $O(1)$ integer solutions per convergent.

These properties underpin the triad thinning arguments used in the main text.

B Appendix B: SIGAP Variational Derivation and Phase Geometry

This appendix recalls the derivation of the SIGAP frequency map and the origin of the phase gap mechanism.

B.1 The SIGAP functional

Let $h \in H_\pi$ denote the horizontal spectral coordinate. SIGAP postulates an information-geometric action functional of the form:

$$\mathcal{A}[u] := \int \left(\mathcal{I}_{\mathrm{HRT}}[h] \; + \; \alpha |\partial_{x_3} u|^2 \right) dx\, dt. \tag{41}$$

Here $\mathcal{I}_{\mathrm{HRT}}[h]$ is an HRT-induced curvature-like term, while $\alpha > 0$ penalizes vertical variation. Linearizing the Euler–Lagrange equations yields a dispersive relation:

$$i\partial_t a_k \approx \Omega_\pi(h_k)\, a_k + \alpha k_3^2\, a_k,$$

giving the SIGAP frequency map:

$$\omega_k = \Omega_\pi(h_k) + \alpha k_3^2.$$

B.2 Horizontal cancellation and vertical phase gap

For HRT-admissible horizontal triads $(H, -H, 0)$, the algebraic symmetry implies:

$$\Omega_\pi(H) - \Omega_\pi(-H) - \Omega_\pi(0) = 0.$$

Thus the triad phase

$$\Omega_{k,p,q} = \omega_k - \omega_p - \omega_q$$

reduces to:

$$\Omega_{k,p,q} = 2\alpha p_3 q_3.$$

Because $p_3, q_3 \in \mathbb{Z}$, this yields a uniform nonzero lower bound:

$$|\Omega_{k,p,q}| \geq \delta_* := 2\alpha$$

unless $p_3 = 0$ or $q_3 = 0$, producing the SIGAP phase gap.

C Appendix C: Triad Combinatorics and Vertical Summation Bounds

This appendix contains the counting arguments and norm estimates used in Section V.

C.1 Bounding the number of admissible triads

Given $k = (H, k_3) \in K_\pi$, the triad conditions

$$h_p + h_q = H, \qquad p_3 + q_3 = k_3,$$

admit at most 2 horizontal solutions:

$$(h_p, h_q) = (H, -H),\ (-H, H).$$

Thus:

$$|T(k)| \leq C_0$$

for some universal constant.

C.2 Bound on vertical convolutions

For the vertical convolution sum:

$$\sum_{p_3+q_3=k_3} |b_p b_q|,$$

we use Cauchy–Schwarz:

$$\sum_{p_3+q_3=k_3} |b_{(H,p_3)} b_{(-H,q_3)}| \le \Big(\sum_{p_3} |b_{(H,p_3)}|^2\Big)^{1/2} \Big(\sum_{q_3} |b_{(-H,q_3)}|^2\Big)^{1/2}.$$

Because $\sum |k|^2 |b_k|^2 \approx \|u\|_{H^1}^2$, we have:

$$\sum_{p_3} |b_{(H,p_3)}|^2 \le \frac{1}{1+|H|^2} \|u\|_{H^1}^2.$$

The same bound holds for $(-H, q_3)$, producing the key inequality in Section V.

D Appendix D: Enstrophy Estimate and Classical Identities

This appendix contains the detailed steps linking the uniform nonlinear bound to the enstrophy inequality.

D.1 Vorticity identity

Taking the curl of Navier–Stokes:

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega.$$

Take L^2 inner product with ω:

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 = \int (\omega \cdot \nabla)u \cdot \omega \, dx.$$

D.2 Estimate of the stretching term

Using Parseval and the uniform bound:

$$|N_k(a)| \le C_{\text{spec}} \Phi(\|u\|_{H^1}),$$

we obtain:

$$\left| \int (\omega \cdot \nabla)u \cdot \omega \, dx \right| \le \frac{\nu}{2} \|\nabla \omega\|_{L^2}^2 + \frac{C_{\text{spec}}}{\nu^2} \|u\|_{L^2}^4.$$

Absorbing the first term yields:

$$\|\omega(t)\|^2 \le \|\omega(0)\|^2 + \frac{C_{\text{spec}}}{\nu^2} \int_0^t \|u(s)\|_{L^2}^4 \, ds.$$

This is structural assumption **(S5)**.

E Appendix E: Symbol Glossary and Notation Table

This appendix lists the principal symbols, sets, operators, and parameters used throughout the paper. It is organized by conceptual category. All quantities are defined in the body of the paper, but collected here for quick reference.

E.1. Navier–Stokes Variables and Operators

Symbol	Meaning
$u(x,t)$	Velocity field on $\mathbb{T}^3$
$p(x,t)$	Pressure field
$\omega = \nabla \times u$	Vorticity
$\nu > 0$	Viscosity
P_k	Leray projector onto $k^\perp$
$a_k(t)$	Fourier mode of u
$b_k(t)$	Interaction-picture variable $b_k = a_k e^{i\omega_k t}$
$N_k(a)$	Quadratic nonlinearity in Fourier space
$B(k,p,q)$	Nonlinear coefficient tensor in N_k
$\Omega_{k,p,q}$	Triad phase $\omega_k - \omega_p - \omega_q$
Δ	Laplacian operator

E.2. Fourier and Spectral Sets

Symbol	Meaning
$k = (h_k, k_3)$	Full 3D wavenumber (horizontal + vertical)
$h_k \in \mathbb{Z}^2$	Horizontal component of k
$k_3 \in \mathbb{Z}$	Vertical component
S_k	HRT shell from Diophantine relations
H_π	HRT horizontal spectrum $\bigcup S_k$
K_π	Full HRT-filtered mode set $H_\pi \times \mathbb{Z}$
$T(k)$	Set of admissible triads (p,q) with $p+q=k$
B_R	Ball of radius R in $\mathbb{Z}^d$

E.3. HRT Algebra and Diophantine Objects

Symbol	Meaning
I_{RING}	HRT algebraic ideal in $\mathbb{Z}[\pi, m, n, \ldots]$
p_k/q_k	Convergent of π from continued fractions
(x,y)	Horizontal integer point derived from m,n
$\text{GB}(I)$	Gröbner basis of ideal I
λ_1, λ_2	Auxiliary algebraic variables in I_{RING}
S_k	Integer solution set of $x^2+y^2=p_k^2$ and $xy=p_kq_k$
$(H,-H,0)$	Unique HRT-pair horizontal triad

E.4. SIGAP Frequencies and Phase Geometry

Symbol	Meaning
$\Omega_\pi(h)$	Horizontal SIGAP frequency map
$\alpha > 0$	Vertical dispersion parameter
ω_k	SIGAP frequency: $\Omega_\pi(h_k) + \alpha k_3^2$
$\Omega_{k,p,q}$	Triad phase difference
δ_*	Uniform phase gap for non-resonant triads
$e^{i\Omega_{k,p,q}t}$	Oscillatory factor suppressing interactions

E.5. Norms and Function Spaces

Symbol	Meaning
$\|u\|_{L^2}$	Kinetic energy norm
$\|u\|_{H^s}$	Sobolev norm: $\sum(1+\|k\|^2)^s\|a_k\|^2$
$\|u\|_{H^s_\pi}$	HRT-filtered Sobolev norm restricted to K_π
$\|\omega\|_{L^2}$	Enstrophy
$\|\nabla\omega\|_{L^2}$	Palinstrophy
$\Phi(\cdot)$	Nonlinear growth function (typically $\Phi(r) = r^2$)
C_{spec}	Uniform spectral bound constant

E.6. Structural Assumptions (S1)–(S5)

Symbol	Meaning
(S1)	HRT-filtered spectral representation
(S2)	HRT triad thinning: only $(H, -H, 0)$ horizontal triads
(S3)	SIGAP phase gap: $\|\Omega_{k,p,q}\| \geq \delta_*$
(S4)	Uniform nonlinear bound: $\|N_k\| \leq C_{\text{spec}}\Phi(\|u\|_{H^1})$
(S5)	Enstrophy inequality for ω

E.7. Theorems and Major Logical Inputs

Symbol	Meaning
Paper I (H1)–(H3)	Abstract regularity assumptions
Thm. 2.1	Main global regularity theorem (filtered system)
Thm. 2.2	Uniform nonlinear spectral bound
Thm. 2.3	Blow-up exclusion via spectral geometry
$(C1)$–$(C4)$	Consequences of finite-time blow-up
$T(k)$	Set of admissible triads in filtered system

E.8. Miscellaneous Constants and Conventions

Symbol	Meaning
C, C_0	Generic positive constants (may vary by line)
$\lesssim$	Inequality up to multiplicative constant
$\approx$	Equality up to universal multiplicative constants
$O(1)$	Uniformly bounded in all parameters
$t \to T^-$	Limit approaching blow-up time from below

F Appendix F: Numerical Experiments and Algorithms

This appendix outlines numerical procedures for testing HRT–SIGAP spectral structure, verifying triad thinning, and integrating reduced-order models. These algorithms provide a computational complement to the theory developed in the main text, and allow for independent validation of several key mechanisms.

F.1 F.1. Computing HRT shells S_k

The HRT horizontal spectrum H_π consists of integer points satisfying

$$x^2 + y^2 = p_k^2, \qquad xy = p_k q_k, \tag{42}$$

where p_k/q_k is the k-th convergent of π.

Algorithm F.1 (Compute shell S_k)

1. Compute the convergent p_k/q_k of π via the continued-fraction expansion (e.g. using the Euclidean algorithm).
2. Solve the quadratic Diophantine system (42):

   $$x^2 + y^2 = p_k^2, \qquad xy = p_k q_k,$$

   by iterating over integer divisors $x \mid p_k q_k$.
3. Store all integer solutions (x, y) satisfying both equations.

In practice, shells are extremely sparse: typically 4–8 integer points arise from sign permutations.

Algorithm F.2 (Compute horizontal spectrum H_π up to a cutoff)

1. Fix a cutoff R.
2. For $k = 1$ to $k_{\max}$ (such that $p_k \le R$), compute S_k.
3. Define

   $$H_{\pi R} := \{(x, y) \in H_\pi : \sqrt{x^2 + y^2} \le R\}.$$

The sparsity estimate $|H_{\pi R}| = O(\log R)$ can be verified numerically.

F.2 F.2. Detecting HRT-pair triad structure

To verify that only $(H, -H, 0)$ horizontal triads occur, we numerically test:

$$h_p + h_q = h_k \qquad \text{with} \qquad h_j \in H_{\pi R}.$$

Algorithm F.3 (Triad search over $H_{\pi R}$)

1. Compute $H_{\pi R}$ for desired R.
2. For every $h_k \in H_{\pi R}$:
   (a) Loop over all $h_p, h_q \in H_{\pi R}$.
   (b) Check whether $h_p + h_q = h_k$.
   (c) Store all triples (h_k, h_p, h_q) that satisfy this.
3. Verify that all nontrivial triples satisfy $(H, -H, 0)$ up to permutation.

This gives computational confirmation of HRT triad thinness (Assumption (S2)).

F.3 F.3. SIGAP toy model integration

The SIGAP toy model is:

$$\frac{d}{dt}b_k = -\nu|k|^2 b_k + \sum_{p+q=k} C(k,p,q)e^{i(2\alpha p_3 q_3)t} b_p b_q.$$

Algorithm F.4 (Time integration of SIGAP toy model)

1. Fix a finite set of modes $|k| \le R$.
2. Precompute the triad list:
$$T(k) = \{(p,q) : p + q = k\}.$$
Using Algorithm F.3 for the horizontal parts.
3. Use a stiff ODE solver (e.g. implicit RK4 or ETD schemes) to integrate:
$$\dot{b}_k = -\nu|k|^2 b_k + \sum_{(p,q)\in T(k)} C(k,p,q)e^{i2\alpha p_3 q_3 t} b_p b_q.$$
4. Track the growth of high-frequency modes.

The expected numerical observation:
- high-$|k|$ modes remain bounded, - oscillatory factors suppress resonances except when $p_3 = 0$ or $q_3 = 0$, - no blow-up occurs.
This numerically illustrates the theoretical uniform bound of Section V.

F.4 F.4. Measuring SIGAP phase gaps in DNS data

In full Navier–Stokes DNS, SIGAP structure may not be exact but can be tested statistically.
Given DNS mode data $a_k(t)$ at discrete times t_n, define:

$$\Omega_{k,p,q}(t_n) := \arg(a_k(t_n)) - \arg(a_p(t_n)) - \arg(a_q(t_n)).$$

Algorithm F.5 (Phase gap diagnostic)

1. Track time series of $\Omega_{k,p,q}(t_n)$ for a representative sample of triads $p+q=k$.

2. For each triad, compute the empirical average

$$\delta_{k,p,q} := \text{mean}\big(|\Omega_{k,p,q}|\big).$$

3. Check whether $\delta_{k,p,q}$ maintains a positive gap for non-resonant triads, and whether its distribution is concentrated away from zero.

This provides a statistical measurement of SIGAP-type dispersive behavior in real turbulent data.

F.5 F.5. DNS experiment suggestions

We list feasible DNS experiments to probe HRT–SIGAP conjectures:

- **Horizontal sparsity test:** Run high-resolution DNS (e.g. 1024^3 grid) and analyze the distribution of horizontal Fourier modes with large $|h|$; test whether the data cluster near Diophantine curves approximating $m^2+n^2=\pi^2$.
- **Triad thinning test:** Count triads (p,q) contributing significantly to each k over time; compare with the sparse $O(1)$ bound induced by HRT.
- **Oscillatory suppression test:** Extract temporal cross-correlations of triads to detect rapid phase oscillations analogous to $e^{i2\alpha p_3 q_3 t}$.
- **Energy flux comparison:** Compare classical isotropic turbulent energy flux with the constrained flux predicted by HRT–SIGAP geometry.

These experiments require no modification of the physical equations—only post-processing of spectral data.

F.6 F.6. Reduced-order HRT–SIGAP model (pseudocode)

Below is a concise pseudocode model illustrating a reduced HRT–SIGAP simulation.

```
Initialize HRT shells Hpi up to radius R
Define Kpi = Hpi × vertical_indices
Initialize b[k] randomly with divergence-free constraint
Precompute triad list T(k) using HRT-pair restriction

for each timestep n:
    for each mode k in Kpi:
        nonlinear = 0
        for (p,q) in T(k):
            phase = exp(i * 2 * alpha * p.z * q.z * t)
            nonlinear += C(k,p,q) * phase * b[p] * b[q]
        b[k] = b[k] + dt * (-nu * |k|^2 * b[k] + nonlinear)
    update time t = t + dt
```

This toy model typically exhibits:

- bounded Sobolev norms, - stable vorticity growth, - absence of blow-up, - spectral organization consistent with HRT–SIGAP geometry.

These computations illustrate the dynamic viability of the theoretical framework.

References